Order Number 9320754

Transfer and Steenrod squares

Litvanyi, Peter Ivo, Ph.D.

University of Alaska Fairbanks, 1993

U·M·I
300 N. Zeeb Rd.
Ann Arbor, MI 48106

TRANSFER AND STEENROD SQUARES

A

THESIS

Presented to the Faculty

of the University of Alaska Fairbanks

in Partial Fulfillment of the Requirements

for the Degree of

DOCTOR OF PHILOSOPHY

By

Peter Ivo Litvanyi, Univ. Math. Dipl.

Fairbanks, Alaska

May 1993

TRANSFER AND STEENROD SQUARES

By

Peter Ivo Litvanyi

RECOMMENDED:

Robert Piacenza
Advisory Committee Chair

Department Head

APPROVED:

Dean, College of Liberal Arts

Edward
Dean of the Graduate School

27 april 1993
Date

ABSTRACT

Commutators between the transfer and Steenrod squares have been investigated by several authors. Let X be a finite simplicial complex and τ be a regular involution on X. If τ has no fixed point, then the commutator is trivial by certain results in generalized cohomology theory. For involutions with possible fixed points, the commutator was first expressed by Bott as $\Delta^*Sq^i + Sq^i\Delta^* = \mu Sq^{i-1}\Delta^*$. Here Δ^* is the transfer map and μ denotes the connecting morphism of the Smith sequence. Another formula, closely related to the one above, was given by Kubelka and gives the commutator in terms of the cohomology class restricted to the fixed point set and certain characteristic classes arising from the double cover of the complement to the fixed point set. In this thesis, I prove the generalization of the formulas above for sheaf cohomology. As one of the consequences, due to the powerful nature of sheaf theory we gain the results without serious restrictions on the space: X is required to be paracompact, Hausdorff.

In Chapter 1, I review the standard sheaf-theoretical constructions for both the transfer and the Steenrod powers based on Bredon's results. I state and prove a few technical lemmas on Smith sequences that are necessary in my setting.

In Chapter 2, I state and prove the analogue of Bott's formula for paracompact Hausdorff spaces.

In Chapter 3, we derive a generalization of Kubelka's formula for spaces as above.

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Notation List

SPACES AND MAPS

$X, Y, Z, \ldots, X', Y', Z'$	paracompact Hausdorff spaces
$e, f, g, h, \ldots$	continuous maps
$F, L, N, \ldots$	closed subsets
$U, V, \ldots$	open subsets

SPACES WITH INVOLUTION

a, a'	involution *see 1.5.1*
$X/a = Y$	quotient space *see 1.5.1*
$p : X \to Y$	quotient map *see 1.5.1*
L	fixed - point set *see 1.5.1*
$\ell = p(L),\ n = p(N), \ldots$	*see 1.5.1, 1.5.3*
$\widetilde{F}, \widetilde{\ell}, \ldots$	complements *see 1.5.3*

SHEAVES AND MORPHISMS OF SHEAVES

$\mathscr{A}, \mathscr{B}, \ldots$	sheaves ***see [Bredon 1]***
$\mathscr{A}\vert_x, \ldots$	stalks ***see [Bredon 1]***
$\alpha, \beta, \gamma, \ldots$	homomorphisms of sheaves ***see [Bredon 1]***
$f : \mathscr{B} \to \mathscr{A}$	f - cohomomorphism of sheaves ***see [Bredon 1]***
f - Cohom$(\mathscr{B}, \mathscr{A})$	set of f - cohomomorphisms ***see [Bredon 1]***

$\mathrm{Hom}(\mathscr{A}, \mathscr{B})$	set of sheaf homomorphisms *see [Bredon 1]*
$\overleftarrow{f}\mathscr{A}$	the pullback sheaf *see [Bredon 1]*
$f^* : \mathscr{A} \to \overleftarrow{f}\mathscr{A}$	the canonical cohomomorphism to the pullback *see [Bredon 1]*
$\vec{f}\mathscr{A}$	the pushforward sheaf *see [Bredon 1]*
$f^+ : \vec{f}\mathscr{A} \to \mathscr{A}$	the canonical cohomomorphism from the pushforward *see [Bredon 1]*
$\overleftrightarrow{f}\mathscr{A}$	$\vec{f}(\overleftarrow{f}\mathscr{A})$
$\nabla : \mathscr{A} \to \overleftrightarrow{f}\mathscr{A}$	*see 1.2.2*

PROTOSHEAVES

$\mathrm{Germ}^+(n)$, $\mathrm{Germ}(n)$, $\mathrm{Germ}^0(n)$	*see 1.3.3*
$\{\gamma\}$, ...	maps of protosheaves *see 1.8.10*

RESOLUTIONS AND COHOMOLOGY

$\mathscr{M}^*$, $\mathscr{I}^*$, ...	differential graded sheaves / resolutions *see [Bredon 1]*
d	chain differential *see [Bredon 1]*
ε	augmentation *see [Bredon 1]*
η_x , D_x	pointwise splittings *see 1.8.3*
Γ	general section functor *see [Bredon 1]*
D	chain homotopy *see [Bredon 1]*

$\mathscr{C}(X, \mathscr{A})$	canonical flabby resolution *see [Bredon 1]*
$\mathscr{C}^*_x(X,\mathscr{A})$	sections of the canonical flabby resolution *see [Bredon 1]*
$\mathscr{L}^*(X, \mathscr{A})$	*see 1.4.1*
$\mathscr{C}(\underline{f}) = \bar{f}$, $\mathscr{C}(\underline{\gamma}) = \bar{\gamma}$	canonical flabby extensions of (co)homomorphisms *see [Bredon 1]*
$H^*(X, \mathscr{A})$	sheaf cohomology *see [Bredon 1]*
convention 1	The support system used throughout this work is the collection of all closed sets; all symbols above understood in this sense.
convention 2	Maps induced by (co)homomorphisms on cohomology level are denoted by the same letters as the maps themselves.

RELATIVE COHOMOLOGY

$H^*(X, F, \mathscr{A})$	relative cohomology *see [Bredon 1]*
ι_F, ρ_F	*see 1.5.17*
R_F	*see 1.5.17*
δ_F	connecting morphism of the long exact relative cohomology sequence *see [Bredon 1]*

GODEMENT REPRESENTATION

$A^+(n)$, $A(n)$, $\underline{A}(n)$, $A^0(x, n)$	*see 1.3.2*
F	*see 1.4.3*

SMITH SEQUENCES

α	switch homomorphism *see 1.5.9*
σ, σ_N	symmetrization homomorphism *see 1.5.9*
S, S_0, S_N	Smith sequences *see 1.5.9, 1.5.19*
μ, μ_N	connecting morphism of the Smith sequence *see 1.5.9, 1.5.19*
Δ, Δ_N	transfer map *see 1.5.9, 1.5.19*

TENSOR PRODUCT

$\otimes$	generic tensor product of resolutions, cohomomorphisms, etc. *see **[Bredon 1]***
$d_\otimes$	differential on the tensor product *see 1.7.1*
ν	natural "switch" *see 1.7.2*
τ	"symmetrization" *see 1.7.2*
Λ_x, Λ_x^1	pointwise splittings for the tensor product *see 1.8.6, 1.8.7*
θ	diagonal map *see 1.7.4*

STEENROD SQUARES / POWERS

$\langle \chi_k \mid k = 0, 1, \ldots \rangle$	*see 1.7.3, 1.8.11*
St_k	lower Steenrod operations *see 1.7.16*
Sq^k	Steenrod squares *see 1.7.16*
St^k	Steenrod p - powers ($p \neq 2$) *see Appendix B*
ψ_k	*see 1.8.15*

CUP PRODUCT

$\cup$	*see 3.1.1*

MAYER - VIETORIS

D	connecting morphism of the Mayer - Vietoris sequence *see 3.1.5*
P_j^* , Q_j^* , P_j , Q_j	natural transformations *see 3.1 - 3.4*

Acknowledgements

I am very grateful to my advisor, Dr. Robert Piacenza, for his guidance in the preparation of this thesis. Our discussions were a great help, as were his critiques of the many drafts of the manuscript. I would also like to thank Caren Litvanyi for her assistance with the typing.

CHAPTER I

1.1 INTRODUCTION

In this section we give a brief summary of the results that gave motivation to the problems discussed in this thesis. We also give a section-by-section outline.

1.1.1 STEENROD SQUARES

Broadly speaking, algebraic topology concerns functors from a sufficiently convenient topological category to some algebraic category. Perhaps one of the oldest examples of such a functor is the concept of simplicial cohomology; here we assign a graded group $H^*(K, G)$ to a pair (K, G) where K is a finite simplicial complex and G is an abelian group.

It was soon noticed that if $G = R$ is a ring, then $H^*(K, R)$ possesses a ring structure with the multiplication called "cup product". The advantages of working in this stricter algebraic category quickly became evident. The extra structure not only made the computation of cohomology groups easier, but also revealed numerous fine obstructions that in the original group setting had gone undetected. Moreover, the ring structure has an easy and extremely geometrical description on cochain level. A very clear account of the ideas above may be found in **[Munkres]**.

It was the above-mentioned geometrical construction of the cup product that supplied the motivation to N. Steenrod in 1947 to introduce a family of operators $\left\langle Sq^k : H^*(K, \mathbb{Z}_2) \to H^{*+k}(K, \mathbb{Z}_2) \mid k = 0, 1, \dots \right\rangle$ that was later named *the system of Steenrod squares* in his honor. These operators are natural with respect to maps of complexes and can be viewed as elements of a formal algebra $\mathcal{A}(2)$. An alternative way to express this result is to say that $H^*(K, \mathbb{Z}_2)$ has a natural $\mathcal{A}(2)$ - module structure. Other known structures, such as cup product and Bockstein map, turned out to be merely manifestations of this module structure. Steenrod squares and their generalizations for $\mathbb{Z}_p$ coefficients (Steenrod p - powers) combined with the appearance of spectral sequence techniques have developed into a powerful tool in the hands of the contemporary algebraic topologist, and their significance can hardly be over-appreciated.

Steenrod's original paper **[Steenrod 1]** is written very much in the spirit of the forties and is based on complicated combinatorial arguments. As interest shifted towards the new discovery, different machinery had to be developed to handle the upcoming problems and to simplify existing results. A new approach was introduced in **[Steenrod 2]** and finalized in **[Steenrod - Epstein]**. Based on the cohomology of groups, one of the most prominent features of this approach is its strongly algebraic nature. Two facts were clearly revealed in the exposition. First, Steenrod operations are some sort of algebraic symmetrizations on cochain level, and thus they reside in the realm of homological algebra rather than topology. An ultimate example of how elucidating this point of view can be is the derivation of Steenrod squares as in **[Spanier]**. Second, Steenrod operations are inherently connected with another kind of symmetrization process that manifests itself in the Smith - Richardson sequence (see below). For Steenrod squares, this correspondence was investigated in an early paper of R. Bott **[Bott]**. Among numerous other results, he derives an elegant formula describing how these two different kinds of symmetrization commute. His formula has been of considerable motivation to us and will be cited below. An analogous investigation for general primes was carried out by M. R. Thom and W.-T. Wu in **[Thom - Wu]** and **[Wu]**.

1.1.2 TRANSFER AND SMITH - SEQUENCE

The earliest definition of transfer was formulated in the context of covering spaces. Let $p : X \to Y$ be an n - sheeted covering projection. By "lifting" singular simplices and then taking the summation, we can easily obtain a map of singular cochain complexes $\Delta : C^*(X) \to C^*(Y)$. Then, in turn, we induce the transfer map $\Delta : H^*(X) \to H^*(Y)$ on cohomology level. For double covers, the transfer map is an ingredient of the well-known Thom - Gysin sequence. A clear account of the Thom - Gysin construction can be found in **[Spanier]** and we refer the reader there.

Various attempts have been made to construct the transfer map in less restrictive settings. The "Atiyah transfer" **[Atiyah]** suggests that we might define the transfer for generalized cohomology theories even if an "ad-hoc" geometrical construction is not available. This line was investigated by Dold, Becker, and Gottlieb who established the existence of transfer for fiber bundles $p : E \to B$ with smooth compact fiber and Lie structure group **[Becker & Gottlieb]**. The generalization relevant to this thesis takes just the opposite path. It stresses group action rather than fibration properties and, ultimately, sheaf theory rather than generalized cohomology theory.

Let G be the group of deck transformations acting freely on the regular covering projection $p : X \to Y = X/G$ as above. Then the transfer is interpreted as a map $\Delta : H^*(X) \to H^*(X/G)$. This point of view carries over to more general cases. If G is a finite group acting regularly on a simplicial complex X, we can still "lift" the individual simplices in the factor complex X/G back to X. This fact, just as in the covering case, yields a map of the simplicial cochain complexes which generates the transfer map $\Delta : H^*(X) \to H^*(Y)$ where $Y = X/G$. If $G = \mathbb{Z}_p$, there is a generalization of the Thom - Gysin sequence in this context. The above defined transfer is part of a long exact sequence called the Smith - Richardson sequence. Using Čech cohomology and simplicial approximation, the ideas above extend to general paracompact spaces. For the original reference, we cite **[Richardson - Smith]** . For an updated account of Smith theory we refer the reader to **[Bredon 2]** and **[Kawakubo]** .

1.1.3 BOTT'S COMMUTATOR FORMULA

We summarize the relevant result from **[Bott]** here.

Let K be a finite simplicial complex. Let $X := K \times K$ and $a : X \to X$ be defined by $a(k_1, k_2) = (k_2, k_1)$, i.e. the natural $\mathbb{Z}_2$ - action on the product.

Let $Y := X/a$ be the quotient space called the symmetric product. It is well known that $X \times X$ can be given a simplicial decomposition such that a becomes a regular simplicial map. See, for example, **[Eilenberg - Steenrod]** .

Smith theory gives the following long exact sequences:

$$S_0: \quad \cdots \to H^n(X, \mathbb{Z}_2) \xrightarrow{\Delta_0} H^n(Y, \ell, \mathbb{Z}_2) \xrightarrow{\mu_0} H^{n+1}(Y, \mathbb{Z}_2) \xrightarrow{p^*} H^{n+1}(X, \mathbb{Z}_2) \to \cdots$$

and

$$S_L: \quad \cdots \to H^n(X, L, \mathbb{Z}_2) \xrightarrow{\Delta_L} H^n(Y, \ell, \mathbb{Z}_2) \xrightarrow{\mu_L} H^{n+1}(Y, \ell, \mathbb{Z}_2)$$

$$\xrightarrow{p^*} H^{n+1}(X, L, \mathbb{Z}_2) \to \cdots$$

where $\quad L := \{(k, k) \mid k \in K\}$ is the diagonal in X,

$\ell := p(L)$, the diagonal viewed as a subset of Y ,

and $\quad \Delta_0$ and Δ_L are the so-called transfer maps.

Then we have the following statement **[Bott]** (Theorem II) :

$$\Delta_L \circ Sq^k + Sq^k \circ \Delta_L = \mu_L \circ Sq^{k-1} \circ \Delta_L \qquad \text{on } H^*(X, L, \mathbb{Z}_2)$$

$$\Delta_0 \circ Sq^k + Sq^k \circ \Delta_0 = \mu_L \circ Sq^{k-1} \circ \Delta_0 \qquad \text{on } H^*(X, \mathbb{Z}_2)$$

The proof uses simplicial calculations on cochain level. The basic idea is that two kinds of symmetrizations -- one coming from the transfer and the other coming from the Steenrod operation -- intertwine in a neat way since one is "external", the other (the transfer) "internal" on chain complex level.

1.1.4 TRANSFER AND CHARACTERISTIC CLASS

The above formula presents us with the question: "what does the commutator between Steenrod squares and transfer measure?". Let us briefly refer back to 1.1.2 here. Suppose S^n stands for a sufficiently high suspension. Whenever the transfer map is actually induced by a stable map $S^n(Y) \to S^n(X)$, it must commute with the Steenrod operation since the latter is natural and respects the suspension isomorphism. The existence of such a map is precisely what was displayed in **[Becker & Gottlieb]** and what made the definition of transfer possible in a generalized cohomology setting. For double covers, a similar argument is presented in **[Adams]** ; or, see **[Piacenza]** . In this sense we can say that the commutator measures how impossible it is to construct a stable map $S^n(Y) \to S^n(X)$ realizing the transfer on cohomology level. Thus returning to the language of group actions, we can see that Steenrod square and transfer commute for fixed-point free involutions. It is therefore logical to expect that for general involutions the commutator must essentially reflect the shape of the fixed point set and the way the fixed point set is imbedded into X and Y . This argument was made explicit in **[Kubelka]** ; we briefly summarize his results here.

Let X be a finite simplicial complex. Let a be a regular involution $a : X \to X$. Let $Y := X/a$ be the factor complex, and $p : X \to Y$ the natural projection. Let L and ℓ denote the fixed point set in X and in Y respectively. Repeating the usual cell lifting, we can define a transfer map $\Delta : H^*(X, Z_2) \to H^*(Y, Z_2)$. Let $\varkappa$ be a regular neighborhood of ℓ and $r : \mathscr{n} \to \ell$ a deformation retraction. Let $N := p^{-1}(n)$ and $\Omega \in H^1(\mathscr{n}\backslash\ell, \mathbb{Z}_2)$ be the characteristic class of the double cover $p|_{N\backslash L} : N\backslash L \to \mathscr{n}\backslash\ell$. Let $i_L : L \to X$ be the natural inclusion. Let $\dot{r} : \mathscr{n}\backslash\ell \to \ell$ be the restriction of the deformation retraction to $\mathscr{n}\backslash\ell$. Let $D_{\mathscr{n}\backslash\ell} : H^*(\mathscr{n}\backslash\ell, \mathbb{Z}_2) \to H^{*+1}(Y, \mathbb{Z}_2)$ be the coboundary of the Mayer - Vietoris sequence associated to the triple $(Y; \mathscr{n}; Y\backslash\ell)$. Under these conditions we have the following theorem as in **[Kubelka]** :

$$[Sq^k \circ \Delta + \Delta \circ Sq^k]x = D_{\mathscr{n}\backslash\ell}\Big[\sum_{j=0}^{k-1} [\,\Omega^{k-j-1} \cup (\dot{r}^* \circ Sq^j \circ i_L^*)(x)]\Big]$$

$\forall \ x \in H^*(X, \mathbb{Z}_2)$.

Apart from some formal computation based on Bott's formula, the heart of his proof is basically a classical duality argument.

1.1.5 THE CONTENT OF THIS PRESENT WORK

The motivation for this thesis was given by the fact that the results in 1.1.4 and 1.1.5 obviously extend to Čech cohomology. Indeed, both Steenrod squares and transfer are defined on paracompact spaces via the usual nerve argument (**[Eilenberg - Steenrod]**, **[Bredon 2]**) . If we generalize the statements above from the finite setting to any simplicial complex, then we can automatically extend the results for Čech cohomology and paracompact spaces with involution. In fact, there is not much to generalize: all arguments work perfectly in the general simplicial setting. Čech cohomology can be regarded as sheaf cohomology with the coefficient being the constant $\mathbb{Z}_2$ - sheaf. Therefore it is natural to ask the following question: "what are the analogues of the theorems above for more general sheaves?" . Direct limit arguments become insufficient to describe this situation; we must go deep into sheaf theory to find the right machinery. By using sheaf methods, completely new proofs of the formulae of both Bott and Kubelka will be given.

Furthermore, these results are established for general coefficient sheaves of $\mathbb{Z}_2$ - algebras and for more general spaces. Here we give a brief description of the work in each main section.

(1.2) We give a short list of basic results we assume are known to the reader. A standard reference is **[Bredon 1]**.

(1.3) & (1.4) In **[Godement]** the author describes a fairly tangible representation of the canonical flabby resolution. It was decided to present his description here in great detail for two reasons. First, his presentation is rather sketchy; it is hoped that the treatment given here in this paper will be more illuminating and to the reader's benefit. Second, his construction does not seem to be available in the English literature. This presentation may be extremely useful for the novice in sheaf theory who would like to see something more tangible than an inductive definition behind the elusive idea of the canonical flabby resolution.

(1.5) In this section we give a precise definition of the transfer map. The definition itself follows along the lines of **[Bredon 1]**. We derive a Smith sequence in the spirit of the reference above and investigate the naturality property for this sequence. We have not found this to have been investigated in sheaf-theoretical context elsewhere.

(1.6) A technical lemma is stated and proved. The simplicial version of this result may be found in [Bott]. Compared to the original the proof we will give is rather complicated. This shows clearly how sheaf coefficients can complicate even the simplest argument. We freely use the Godement representation developed in 1.3 thru 1.4.

(1.7) There is very little original work in this section. It was Bredon who generalized the Steenrod operations in the context of Sheaf theory. A good description of his ideas can be found in **[Bredon 1]**. Unfortunately, this result has not yet become part of the standard graduate algebraic topology fare. One of the possible reasons for this is the somewhat overwhelming exposition in his book which is meant to cope with the most general case. When we work with sheaves of $\mathbb{Z}_2$ - algebras and Steenrod squares, much of the discussion simplifies. We reiterate his construction in this setting. It is hoped that this section might serve as an introduction to the material for the reader who wants to catch a glimpse of Bredon's essential ideas without the heavy technicalities that are necessary for the more general case.

(1.8) We proceed to outline Bredon's ideas. The only original work is Lemma 1.8.13. The significance of this observation is that it makes diagram chasing possible in the sheaf-theoretical context. In a certain sense, this is the result that gives us the key to generalizing Bott's theorem.

(2.1) Having all the ingredients identified, we formulate the sheaf-theoretical version of 1.1.3. The Vietoris - Begle theorem is used to reduce the statement to a certain diagram chasing.

(2.2) We carry out the proof. Although the proof is essentially just an identification of the connecting map in a snake lemma, it is a fairly tricky specimen of its kind. It is based on the same idea as the simplicial analogue, namely that one of the symmetrizations is external while the other is internal.

(2.3) We define a version of the transfer to be used in the generalization of Kubelka's formula. We also derive a few consequences of the Bott formula. Among others, we prove the fact that the transfer and Steenrod square commute for double covers even in the sheaf-theoretical setting. The results from 1.6 enable us to use the same formulas as in **[Kubelka]**, these being formal manipulations that go unchanged in both contexts.

(3.1) We describe the well-known machinery of the Gysin sequence, Mayer - Vietoris argument, and characteristic class in sheaf-theoretical context. A standard reference is **[Bredon 1]**. We prove the equivalence of the Gysin and Smith sequences in sheaf theory in Appendix A. The reason is that the proof uses Čech machinery and it was felt to be somewhat out of context in 3.1. This proof is new; for the untwisted case (constant coefficient) a standard reference is **[Bredon 2]**.

(3.2) We make it clear what we meant in stating that Kubelka's argument was a classical duality argument. We define two natural transformations and prove their equivalence. The method is the same as in **[Kubelka]**.

(3.3) We state our generalization of Kubelka's theorem in terms of the functors above. We also show how his theorem follows from our version. In the process we disposed of some of his overly restrictive conditions such as the space being a finite simplicial complex or the action being regular.

(Concluding remarks)

We state some of the open problems that emerged from the investigations made in this present thesis. Most of them may very well be approachable via the methods outlined in this exposition.

(Appendix A)

We prove the equivalence of Smith and Gysin sequences for double covers and general sheaf coefficients as was mentioned in 3.1. We use the Čech description of sheaf theory as in [Spanier].

(Appendix B)

This appendix answers some questions that may come to the reader's mind while pondering open problem (7) in the concluding remarks. Suppose G is a finite group and p is a prime. We prove that Steenrod p - power and transfer corresponding to a general G - action commute if p is not a divisor of G. This explains why we attribute particular importance to $\mathbb{Z}_p$ - actions rather than general finite group actions.

1.2 PREREQUISITES

In this section we recall a few foundational results in sheaf theory. Given the large number of excellent expository works available on the subject, we will suppose a certain familiarity with the field.

1.2.1 SHEAVES AND PRESHEAVES

The reader is assumed to be familiar with the concepts below:

- definition of and elementary properties of sheaves and presheaves
- algebraic constructions with sheaves; in particular, tensor product of sheaves and quotient by a subsheaf
- definition of the direct image ("push forward") and inverse image ("pull back") sheaf
- definition of cohomomorphism of sheaves

- existence of the natural cohomomorphism between the pull back / push forward sheaf and the original sheaf

For a quick introduction or review we refer the reader to [Bredon 1] chapter 1. In particular, we take the following fact as known.

1.2.2 FACT

Let $f : X \to Y$ be a map. Let $\mathscr{A}$ and $\mathscr{B}$ be sheaves on X and Y respectively. Then we have the following natural isomorphism of functors.

$$\mathrm{Hom}(\vec{f}\mathscr{B}, \mathscr{A}) \cong f\text{-}\mathrm{Cohom}(\mathscr{B}, \mathscr{A}) \cong \mathrm{Hom}(\mathscr{B}, \overleftarrow{f}\mathscr{A}) .$$

Moreover, we have a natural homomorphism

$$\nabla : \mathscr{B} \to \overleftrightarrow{f}\mathscr{B} ,$$

which is injective if f is onto.

<u>Proof</u>:

See [Bredon 1] pp. 10 - 12 . ∎

1.2.3 EXTENSIONS AND RESTRICTIONS

We expect the reader to be familiar with

- the definition of restriction and the existence of the natural cohomomorphism to the restricted sheaf;
- the idea of extension by zero.

Notation:

Let $\mathscr{A}$ be a sheaf on X and let S be a subspace. $\mathscr{A}|_S$ will be used to denote the restricted sheaf on S while $(\mathscr{A})_S$ will stand for $\mathscr{A}|_S$ extended to X by zero.

1.2.4 FACT

Let be a sheaf on X. Let $U \subset X$ be open and $F := X \setminus U$. Then the sequence

$$(\mathscr{A})_U \rightarrowtail \mathscr{A} \twoheadrightarrow (\mathscr{A})_F$$

is exact.

Proof:

See **[Bredon 1]** p. 8. ■

1.2.5 DIFFERENTIAL SHEAVES AND RESOLUTIONS

We also take the following concepts as known.

- differential graded sheaf
- resolution of a sheaf
- construction of the canonical flabby resolution (serration resolution)

For a reference, the reader may see **[Bredon 1]** chapter 2.

1.2.6 FACT

(1). The canonical flabby resolution is exact. Moreover, it is exact on presheaf level.

(2). The canonical flabby resolution is pointwise homotopically trivial.

(3). Cohomomorphisms of sheaves automatically extend to cohomomorphisms of the canonical flabby resolutions.

Proof:

[Bredon 1] pp. 26 - 28. ■

1.2.7 INJECTIVE SHEAVES

The following concepts will also be used freely:

- the idea of injective sheaf; in this thesis the coefficient ring is taken to be the constant sheaf $\mathbb{Z}_2$.
- the fact that the canonical flabby resolution to a sheaf of $\mathbb{Z}_2$ - algebras is an injective resolution.

We refer to **[Bredon 1]** pp. 30 - 31. A well-known result along these lines which we will use is the following.

1.2.8 FACT

Let $\mathscr{A}$ and $\mathscr{B}$ be sheaves on X. Let $\gamma : \mathscr{A} \to \mathscr{B}$ be a homomorphism. Let $\mathscr{M}^*$ be a resolution of and $\mathscr{I}^*$ be an injective resolution of $\mathscr{B}$. Then

(1). γ extends to a map of resolutions, i.e. to a map of differential graded sheaves.

(2). Any two such extensions are chain homotopically equivalent.

(3). If $\mathscr{A} = \mathscr{B}$ and $\gamma = \mathrm{Id}$, then the first chain homotopy $D^1 : \mathscr{M}(X, \mathscr{A}) \to \mathscr{B}$ can be chosen to be zero for any two extensions.

<u>Proof</u>:

See **[Bredon 1]** pp. 31 - 32. ■

1.2.9 CUP PRODUCT

Though it is not absolutely necessary, the reader should be somewhat familiar with the machinery of cup products in sheaf-theoretical context. We again refer to **[Bredon 1]** p. 40.

1.2.10 DIFFERENT RESOLUTIONS

We assume the reader is familiar with the following facts.

- For the computation of cohomology, we do not necessarily have to use the canonical resolution. In fact, any acyclic resolution can be used for this purpose. We use the support system consisting of all closed sets of a paracompact space; "acyclic" above is understood to be with respect to this particular system.
- The extension of a cohomomorphism between two acyclic resolutions (if such an extension is possible) computes the same map on cohomology level as the canonical flabby extension.
- We have the following implications.

  (1). $\mathscr{M}^*$ is an injective resolution implies $\mathscr{M}^*$ is a flabby resolution which in turn implies that $\mathscr{M}^*$ is an acyclic resolution.

  (2). $\mathscr{A}$ is a flabby/injective resolution implies $\vec{f}\mathscr{A}$ is flabby/injective.

1.2.11 VIETORIS - BEGLE THEOREM

Let X be a paracompact space and $f: X \to Y$ be an onto, closed map. Let $\mathscr{A}$ be a sheaf on X. Suppose $H^p(f^{-1}(y), \mathscr{A}|_{f^{-1}(y)}) = 0$ for all $y \in Y$ and $p > 0$. Then the natural cohomomorphism $f^+ : \vec{f}\mathscr{A} \to \mathscr{A}$ induces an isomorphism

$$f^+ : H^*(Y, \vec{f}\mathscr{A}) \to H^*(X, \mathscr{A}) .$$

Here cohomologies are understood to be with the support system consisting of all closed sets. Notice that the conditions obviously hold for the case when $f^{-1}(y)$ is discreet for all $y \in Y$.

<u>Proof</u>:

See **[Bredon 1]** p. 54. For a classical version, see **[Spanier]**. ■

1.2.12 ADDITIONAL TOPICS

Some additional knowledge is desirable on the following topics:

- relative cohomology sequences in general **[Bredon 1]**
- Mayer - Vietoris / excision type arguments in sheaf theory **[Bredon 1]** (we summarize some of the results in 3.1)
- Gysin sequence in sheaf theory **[Bredon 1]** (we give an account of the $\mathbb{Z}_2$ case in Appendix A)
- Čech cohomological interpretation of sheaf cohomology **[Spanier]** (we use this machinery in Appendix A)
- Steenrod p - powers for sheaf theory **[Bredon 1]** (we treat Steenrod squares in detail; see 1.7 and 1.8)
- transfer map for finite group action **[Bredon 1]** (we treat the $\mathbb{Z}_2$ - action case in detail; see 1.5)

1.3 SECTIONAL NEIGHBORHOODS AND GERMS

In this section we develop some machinery that is essential for an explicit description of the canonical flabby resolution. The description itself is found in section 1.4, and is used in further computations such as in the proof of Lemma 1.6.1.

1.3.1 DEFINITION OF THE SECTIONAL SYSTEMS $\mathcal{S}(x, n)$, $\mathcal{S}(U, n)$, and $\mathfrak{R}(x, n)$

Let X be a paracompact space and $n \in \mathbb{N}$, the natural numbers. Let $\mathcal{P}$ stand for the power set operation. Define for $x \in X$ and $U \subseteq X$, open:

$$\mathcal{S}(x, 0) := \{ U_x \mid U_x \text{ is an open neighborhood of } x \}$$

$$\mathcal{S}(U, 0) := \{ U \}$$

$$\mathfrak{R}(x, 0) := \{ x \}$$

Inductively, we can define the systems $\mathscr{S}(x, k)$, $\mathscr{S}(U, n)$, and $\mathscr{R}(x, n) \subset \mathscr{P}(\underset{n+1}{\times} X)$ as

$$\mathscr{S}(x, n) := \{ S \mid S = \underset{y \in U_x}{\cup^*} [y \times S(y)] , S(y) \in \mathscr{S}(y, n\text{-}1) , \text{and } U_x \text{ is an open nbhd of } x \}$$

$$\mathscr{S}(U, 0) := \{ S \mid S = \underset{y \in U}{\cup^*}[y \times S(y)] \text{ where } S(y) \in \mathscr{S}(y, n\text{-}1) \}$$

$$\mathscr{R}(x, n) := [x \times (\underset{n}{\times} X)] \cap \mathscr{S}(x, n) = x \times \mathscr{S}(x, n\text{-}1)$$

Generally speaking, of course, none of the collection above consists of open sets. However, since they are constructed by directional "pasting" of open sets, we will call them sectional neighborhood systems. All three systems above are directed via the natural inclusion of sets.

Obviously,

$$\cap \mathscr{S}(x, n) = \cap \mathscr{R}(x, n) = (x, x, \ldots, x) \in \underset{n+1}{\times} X$$

and $\cap \mathscr{S}(U, n) = \{ (y, y, \ldots, y) \in \underset{n+1}{\times} X \mid y \in U \}$.

1.3.2 DEFINITION OF THE GROUPS $A^+(n)$, $A(n)$, $\underline{A}(n)$, AND $A^0(x, n)$

Let $\mathscr{A}$ be a sheaf on X. For $x \in X$ and $n \in \mathbb{N}$ we define the following sets:

$$A^+(n) := \{ f : \underset{n+1}{\times} X \to \mathscr{A} \mid f(z_0, \ldots, z_n) \in \mathscr{A}|_{z_n} \}$$

$$A(n) := \{ f \in A^+(n) \mid f(z_0, z_0, z_2, \ldots, z_n) = 0 \in \mathscr{A}|_{z_n} \ \forall\, z_0 \in X \text{ and } \forall\, (z_2, \ldots, z_n) \in \underset{n-1}{\times} X \}$$

$$A^0(x, n) := \{ f \in A(n) \mid f(x, z_1, z_2, \ldots, z_n) = 0 \in \mathscr{A}|_{z_n} \ \forall\, (z_1, z_2, \ldots, z_n) \in \underset{n}{\times} X \}$$

Evidently, we have the natural group structures on all three sets above given by the pointwise addition. If there is any other structure on $\mathscr{A}$, it descends on $A^+(n)$, $A(n)$, and $A^0(x, n)$. Naturally, $A^+(n) \supset A(n) \supset A^0(x, n)$.

Let $f \in A(n)$. We say that f *is locally zero at* x if there is a $g \in A^0(x, n)$ and an $R \in \mathscr{R}(x, n)$ such that $f|_R = g$. Clearly f is locally zero at every $y \in U$ iff $\exists$ $S \in \mathscr{S}(U, n)$ such that $f|_S = 0$. Define $\underline{A}(n)$ to be the collection of functions that are locally zero at every point of X. If we wish to make the dependence on the sheaf or the space explicit, we will write (for example) $A(X, \mathscr{A}, n)$ or $A(\mathscr{A}, n)$.

1.3.3 DEFINITION OF $\mathrm{germ}^+(n)$, $\mathrm{germ}(x, n)$, AND $\mathrm{germ}^0(x, n)$

Let G denote any of the groups defined in 1.3.2 for any fixed $x \in X$. The assignment $S \rightarrow \{f|_S \mid f \in G\}$ gives a direct system of groups.

The limit $\varinjlim_{S \in \mathscr{S}(x,n)} \{f|_S \mid S \in \mathscr{S}(U, n)\}$ is going to be called:

$\mathrm{germ}^+(x, n)$ for $G = A^+(n)$,

$\mathrm{germ}(x, n)$ for $G = A(n)$, and

$\mathrm{germ}^0(x, n)$ for $G = A^0(x, n)$

while it is trivial for $G = \underline{A}(n)$.

Obviously, $\mathrm{germ}^0(x, n) \subset \mathrm{germ}(x, n) \subset \mathrm{germ}^+(x, n)$.

We also introduce the following *protosheaves of groups*:

$$\mathrm{Germ}^+(n) := \bigcup_{x \in X}^{*} \mathrm{germ}^+(x, n) ,$$

$$\mathrm{Germ}(n) := \bigcup_{x \in X}^{*} \mathrm{germ}(x, n) , \text{ and}$$

$$\mathrm{Germ}^0(n) := \bigcup_{x \in X}^{*} \mathrm{germ}^0(x, n)$$

with the natural projection map π onto the base X. Dependence on $X, \mathscr{A}$ will be incorporated into the notation as in 1.3.2 if need be.

1.3.4 DEFINITION OF η_x

The natural map $\mathscr{S}(x, n) \to \mathscr{R}(x, n)$ of directed sets is covered by the restriction of functions. Taking the direct limit, we get $\eta_x : \mathrm{germ}^+(x, n) \twoheadrightarrow \mathrm{germ}^+(x, n-1)$. Obviously η_x descends to $\mathrm{germ}(x, n)$ as $\eta_x : \mathrm{germ}(x, n) \to \mathrm{germ}^0(x, n-1)$. The fact that the symbol η_x was also reserved for the homotopy splitting of the canonical flabby resolution is not an accident. In section 1.4, we shall see that the two concepts actually coincide.

1.3.5 DEFINITION OF [ ] AND [, x]

Let $f \in A(n)$. Then for every fixed $x \in X$ the natural map to the direct limit associates to f an element in $\mathrm{germ}(x, n)$. Composing this by η_x, for any fixed x we can associate $[f, x] \in \mathrm{germ}^0(x, n-1)$ to f. Varying x, we get a map:

$$[\ \ ] : A(n) \to \prod_{x \in X} \mathrm{germ}^0(x, n-1) = \mathscr{C}_X(X, \mathrm{Germ}^0(n-1))$$

It is easy to see that [ ] is actually onto. Its kernel consists of all those functions that are locally zero at every point, i.e., $\underline{A}(n)$. Thus,

1.3.6 LEMMA

(1). $\mathscr{C}(X, \mathrm{Germ}^0(n-1)) \cong \mathrm{Germ}(n)$

(2). $\mathscr{C}_X(X, \mathrm{Germ}^0(n-1)) \cong A(n)/\underline{A}(n)$

Proof:

(2) was stated in the previous definition. (1) comes from presheafifying the argument in the proof of (2) and observing that

$$(\varinjlim_{x\in U} A(U, n))/(\varinjlim_{x\in U} \underline{A}(U, n)) \cong \varinjlim_{S\in \mathcal{S}(x,n)} \{A(n)\} \quad \blacksquare$$

1.3.7 REMARK

Let us state the following trivial but useful fact:

(1). The sequence $\mathrm{germ}^0(x, n) \rightarrowtail \mathrm{germ}(x, n) \xrightarrow{\eta_x} \mathrm{germ}^0(x, n-1)$ is exact.

(2). Or, in global form, $\mathrm{Germ}^0(X, n) \rightarrowtail \mathrm{Germ}(X, n) \twoheadrightarrow \mathrm{Germ}^0(X, n-1)$ is exact.

1.4 THE GODEMENT DESCRIPTION

In this section we give a fairly tangible description of the canonical flabby resolution as in [Godement].

1.4.1 GERM DESCRIPTION OF THE FIRST FEW TERMS IN THE CANONICAL FLABBY RESOLUTION

Let $\mathcal{A}$ be a sheaf over X. The first two steps in the construction of the canonical flabby resolution are indicated in the diagram on the following page.

$$
\begin{tikzcd}
\mathscr{A} \arrow[r, shift left, tail, "\varepsilon"] & \mathscr{C}^0(X,\mathscr{A}) \arrow[l, shift left, two heads, "\eta_x"] \arrow[r, two heads, "\partial"] & \mathscr{L}^1(X,\mathscr{A}) \arrow[d, shift right, "\varepsilon"'] & \\
& & \mathscr{C}^1(X,\mathscr{A}) \arrow[u, shift right, two heads, "\eta_x"'] \arrow[d, two heads, "\partial"] & \\
& & \mathscr{L}^2(X,\mathscr{A}) \arrow[r, shift left, tail, "\varepsilon"] & \mathscr{C}^2(X,\mathscr{A}) \arrow[l, shift left, two heads, "\eta_x"]
\end{tikzcd}
$$

Evidently, we have the following bijection of <u>sets</u>:

$$\mathscr{C}^0(X,\mathscr{A})|_x \; = \; \mathrm{germ}^+(x,0)$$

with $\eta_x : \mathrm{germ}^+(x,0) \to \mathscr{A}|_x$ being the evaluation map.

$$\mathscr{L}^1(X,\mathscr{A}) \;\cong\; \bigcup_{x\in X}{}^{*} \ker\eta_x \;\cong\; \mathrm{germ}^0(0)$$

$$\mathscr{C}^1(X,\mathscr{A}) := \mathscr{C}(\mathscr{L}^1(X,\mathscr{A})) \cong \mathscr{C}(X, \mathrm{germ}^0(0)) = \mathrm{Germ}(1) \quad \text{by Lemma 1.3.6.}$$

In this representation, the evaluation map η_x appears as the "restriction with fixing the first variable" defined in 1.3.4. Again,

$$\mathscr{L}^2(X,\mathscr{A}) \;\cong\; \bigcup_{x\in X}{}^{*} \ker\eta_x \;\cong\; \mathrm{Germ}^0(1) \quad \text{by 1.3.7.}$$

Now

$$\mathscr{C}^2(X,\mathscr{A}) := \mathscr{C}(X,\mathscr{L}^2(X,\mathscr{A})) \cong \mathscr{C}(X,\mathrm{Germ}^0(1)) = \mathrm{Germ}(2)$$

and $\eta_x : \mathrm{germ}(x,2) \twoheadrightarrow \mathrm{germ}^0(x,1)$ is given as in 1.3.7.

One may prove the following lemma by induction.

1.4.2 LEMMA

The following bijections hold:

$$\begin{aligned}
\mathscr{C}^n(X, \mathscr{A}) &\cong \text{germ}(n) \\
\mathscr{C}^n(X, \mathscr{A})|_x &\cong \text{germ}(x, n) \\
\mathscr{L}^n(X, \mathscr{A}) &\cong \text{germ}^0(n-1) \\
\mathscr{L}^n(X, \mathscr{A})|_x &\cong \text{germ}^0(x, n-1) \\
\mathscr{C}_X(X, \mathscr{L}^{n-1}(X, \mathscr{A})) &\cong \mathscr{C}^n_x(X, \mathscr{A}) \cong A(n)/\underline{A}(n)
\end{aligned}$$

and $\eta_x : \text{germ}(x, n) \twoheadrightarrow \text{germ}^0(x, n-1)$ is given by 1.3.7. ∎

1.4.3 DESCRIPTION OF ∂ AND $d = \varepsilon \circ \partial$

It is significantly more troublesome to try to identify the effects of ∂ and d, since this identification requires the exact knowledge of ε. That is, the characterization of $\Gamma(X, \mathscr{L}^{n-1}(X, \mathscr{A}))$ within $\mathscr{C}_X(X, \mathscr{L}^{n-1}(X, \mathscr{A})) \cong A(n)/\underline{A}(n)$.

Let $F : \mathscr{A} \to \mathscr{C}_X(X, \mathscr{A})$ be a fixed map with the property that for any $x \in X$ and $a \in \mathscr{A}|_x$ we have

(1). $\{F(a)\} \in \mathscr{C}_X(X, \mathscr{A})$ is continuous in a neighborhood U_a of x;

(2). $\{F(a)\}(x) = a$;

(3). $\{F(a)\}$ is the zero section if $a = 0 \in \mathscr{A}|_x$.

Obviously, such a map is locally uniquely determined.

Let $f(z_0) \in A^+(0)$. Let $[f, x_0] \in \text{germ}^+(x_0, 1)$ be the corresponding germ at a point $x_0 \in X$. To find $\partial[f, x_0] \in \text{germ}^0(x_0, 0)$, we must find $s_{x_0}(z_0) \in A^+(0)$ such that $s_{x_0}(z_0)$ is continuous around $z_0 = x_0$ and $\eta_{x_0}(f(z_0) - s(z_0)) = 0$, i.e. $f(z_0) = s(z_0)$. Given such an s_{x_0}, $\partial[f, x_0] = [f - s_{x_0}, x_0] \in \text{germ}^0(x_0, 0)$.

Put $s_{x_0}(z_0) := \{F(f(x_0))\}(z_0)$; this obviously meets the requirement. We get that $\partial[f, x_0] = [\, f - \{F(f(x_0))\}, x_0\,] \in \mathrm{germ}^0(x_0, 0)$. We have the following two consequences.

(1). For $f(z_0) \in A^+(0)$ viewed as a section $\mathscr{C}^0_X(X, \mathscr{A})$, the function $df \in A(1)$ defined by $df(z_0, z_1) := f(z_1) - \{F(f(z_0))\}(z_1)$ represents the section $d[f] \in \mathscr{C}^1_X(X, \mathscr{A})$.

(2). A function $g \in A(1)$ will represent an element of $\Gamma(X, \mathscr{L}^1(X, \mathscr{A}))$ within $\mathscr{C}_X(X, \mathscr{L}^1(X, \mathscr{A})) \cong A(1)/\underline{A}(1)$ iff for every $x_0 \in X$ there is an element $f \in A(0)$ such that $g(z_0, z_1) - [\, f(z_0) - \{F(f(z_0))\}(z_1)\,] \in \underline{A}(U, 1)$ for some U open neighborhood of x_0 .

Next, let $g(z_0, z_1) \in A(1)$, and let $[g, x_0] \in \mathrm{germ}(x_0, 1)$ be the corresponding germ element. To find $\partial[g, x_0] \in \mathrm{germ}^0(x_0, 1)$ we must find $s_{x_0}(z_0, z_1) \in A(1)$ with $s_{x_0}(z_0, z_1) \in \Gamma(X, \mathscr{L}^1(X, \mathscr{A}))$ around x_0 and $\eta_{x_0}(g - s_{x_0}) = 0$. By the characterization given in (2) above, this means that for any $x_0 \in X$ we must find an $f_{x_0}(z_0) \in A^+(0)$ such that $g(x_0, z_1) - f_{x_0}(z_1) - F(f_{x_0}(x_0))(z_1) = 0 \quad \forall\, z_1 \in X$.

Take $f_{x_0}(z_0) := g(x_0, z_0)$. It is easy to see that this choice meets the requirement. Just as above, we have proved the statements:

(1). For $f(z_0, z_1) \in A(1)$ viewed as a section of $\mathscr{C}^1_X(X, \mathscr{A})$, the function $df \in A(2)$ defined by $df(z_0, z_1, z_2) := f(z_1, z_2) - f(z_0, z_2) + \{F(f(z_0, z_1))\}(z_2)$ represents the section $d[f] \in \mathscr{C}^2_X(X, \mathscr{A})$.

(2). A function $g \in A(2)$ will represent an element of $\Gamma(X, \mathscr{L}^2(X, \mathscr{A}))$ in $\mathscr{C}_X(X, \mathscr{L}^2(X, \mathscr{A})) = \mathscr{C}^2_X(X, \mathscr{A}) \cong A(1)/\underline{A}(1)$ iff for every $x_0 \in X$ there is an element $f \in A(1)$ such that $g(z_0, z_1, z_2) - [\, f(z_1, z_2) - f(z_0, z_2) - + \{F(f(z_0, z_1))\}(z_2)\,] \in \underline{A}(U, 2)$ for some U open neighborhood of x_0 .

1.4.4 LEMMA

(1). For $f(z_0, z_1, \ldots, z_n) \in A(n)$ viewed as a section of $\mathscr{C}^n_X(X, \mathscr{A})$ the function $df \in A(n+1)$ defined by

$$df(z_0, z_1, \ldots, z_{n+1}) := \sum_{i=0}^{n} (-1)^i f(z_0, \ldots, \hat{z}_i, \ldots, z_{n+1}) + (-1)^{n+1}\{F(f(z_0, z_1, \ldots, z_n))\}(z_{n+1})$$

represents the section $d[f] \in \mathscr{C}^{n+1}_X(X, \mathscr{A})$.

(2). A function $g \in A(n+1)$ will represent an element of $\Gamma(X, \mathscr{L}^{n+1}(X, \mathscr{A}))$ in $\mathscr{C}_X(X, \mathscr{L}^{n+1}(X, \mathscr{A})) = \mathscr{C}^{n+1}_X(X, \mathscr{A}) \cong A(n+1)/\underline{A}(n+1)$ iff for every $x_0 \in X$ there is an element $f \in A(n)$ such that

$$g(z_0, z_1, \ldots, z_{n+1}) - [\sum_{i=0}^{n} (-1)^i f(z_0, \ldots, \hat{z}_i, \ldots, z_{n+1}) + (-1)^{n+1}\{F(f(z_0, z_1, \ldots, z_n))\}(z_{n+1})]$$

$\in \underline{A}(U, n+1)$ for some U open neighborhood of x_0.

Proof:

The proof is by induction. For $n = 0, 1$ the validity of the statement has already been shown.

Suppose it is true for $(n-1)$. Let $g \in A(n)$. To find $\partial[g, x_0]$, by the inductive supposition (2) we have to find a continuous modification $f_{x_0} \in A(n-1)$ such that

$$\eta_{x_0}[g(z_0, z_1, \ldots, z_n) - df_{x_0}(z_0, z_1, \ldots, z_n)] = 0$$

or

$$g(x_0, z_1, \ldots, z_n) - [f_{x_0}(z_1, \ldots, z_n) + \sum_{i=1}^{n-1} (-1)^i f_{x_0}(x_0, z_1, \ldots, \hat{z}_i, \ldots, z_n) + (-1)^n\{F(f_{x_0}(x_0, z_1, \ldots, z_{n-1}))\}(z_n)] = 0$$

$\forall\ (z_1, \ldots, z_n) \in \times_{n-1} X$.

The choice $f_{x_0}(z_0, z_1, ..., z_{n-1}) := g(x_0, z_0, z_1, ..., z_{n-1})$ meets the requirement obviously. If we vary x_0 as the variable z_0 , i.e. take the image of the section [g] , we get that the function

$$\begin{aligned} dg(z_0, z_1, ..., z_{n+1}) := {} & g(z_1, ..., z_{n+1}) - g(z_0, z_2, ..., z_{n+1}) \\ & + \sum_{i=2}^{n} (-1)^i g(z_0, z_1, ..., \hat{z}_i, ..., z_{n+1}) \\ & + (-1)^{n+1}\{F(g(z_0, z_1, ..., z_n))\}(z_{n+1}) \\ = {} & \sum_{i=0}^{n} (-1)^i f(z_0, ..., \hat{z}_i, ..., z_{n+1}) + (-1)^{n+1}\{F(f(z_0, z_1, ..., z_n))\}(z_{n+1}) \end{aligned}$$

which is our statement (1) for n . Statement (2) immediately follows, since the continuous sections of the factor sheaf are locally images of the section space we factored by a subsheaf. ■

1.4.5 SUMMARY

The chain complex $\{\mathscr{C}^*_X(X,\mathscr{A}), d\}$ is isomorphic to the chain complex $\{A(n)/\underline{A}(n), d\}$ where d for the second chain complex is defined as

$$\begin{aligned} df(z_0, z_1, ..., z_{n+1}) := {} & \sum_{i=0}^{n} (-1)^i f(z_0, ..., \hat{z}_i, ..., z_{n+1}) \\ & + (-1)^{n+1}\{F(f(z_0, z_1, ..., z_n))\}(z_{n+1}) . \end{aligned}$$

1.4.6 OBSERVATION

Notice that cohomomorphisms are particularly easy to describe in terms of the complex $\{A(n)/\underline{A}(n), d\}$. Let $k : X \to Y$ and $\underline{k}$ be a cohomomorphism covering k between the sheaves $\mathscr{A}$ on X and $\mathscr{B}$ on Y . Let $f \in A(Y, \mathscr{B}, n)$. Define $(\underline{k}f) \in A(Y, \mathscr{B}, n)$ by

$$(\underline{k}f)(z_0, z_1, ..., z_n) := (\underline{k}|_{k(z_n) \to z_n}) \circ f(k(z_0), k(z_1), ..., k(z_n)) .$$

Then $f \to \underline{k}f$ gives us the chain map induced by the cohomomorphism $\underline{k}$. It is easy to see that this construction respects the composition of cohomomorphisms.

1.4.7 NOTE

Let us note the following trivial but useful fact. Let $\mathscr{A}$ and $\mathscr{B}$ be sheaves on X and $\iota : \mathscr{A} \rightarrowtail \mathscr{B}$ an injection. Then ι induces a map $\iota : A(X, \mathscr{A}, n) \to A(Y, \mathscr{B}, n)$ as above. Obviously, $f \in \underline{A}(X, \mathscr{A}, n)$ iff $\iota \circ f \in \underline{A}(X, \mathscr{B}, n)$; and in particular,

$\iota : A(X, \mathscr{A}, n)/\underline{A}(X, \mathscr{A}, n) \to A(Y, \mathscr{B}, n)/\underline{A}(X, \mathscr{B}, n)$ is injective. This is just the manifestation of the facts that the functor $\mathscr{C}^*$ is exact and the section functor is left exact.

1.5. SMITH SEQUENCES OF AN INVOLUTION

In this section, we give a definition of the transfer map and derive a Smith sequence in the spirit of [Bredon 1]. We will also investigate the naturality properties of this sequence.

1.5.1 NOTATION

Let the symbol (X, a) denote a paracompact Hausdorff space X together with an involution; i.e., a map $a : X \to X$ such that $a \circ a = Id$. Let $Y := X/a$ be the quotient space; $p: X \to Y$ will stand for the natural projection.

If we introduce L for the fixed point set of a, then L can be viewed as a subset of Y just as well via the map p. If we choose to view L in this manner (as a subset of Y), we will use the notation $\mathscr{L}$.

We have the following standard observation:

1.5.2 OBSERVATION

p is open, closed, and proper. Y is paracompact, Hausdorff.

L is closed in X. $\mathscr{l}$ is closed in Y.

See **[Kawakubo]** for details.

1.5.3 NOTATION

Let $N \subset X$ be a closed, invariant subset and $\mathscr{n} := p(N)$. Then $\mathscr{n}$ is closed in Y. Let "~" denote the appropriate complement; e.g., $\widetilde{N} := X\backslash N$ and $\widetilde{\mathscr{n}} := Y\backslash\mathscr{n}$.

In what follows, we will rely heavily on the next lemma.

1.5.4 LEMMA

Let (X, a), (X', a') be spaces with involution. Let $\mathscr{A} \to X$ and $\mathscr{A}' \to X'$ be sheaves. Let $e : X \to X'$ be an equivariant map and let $\underline{e} : \mathscr{A}' \to \mathscr{A}$ be a cohomomorphism covering e. Denote by $[e] : Y \to Y'$ the induced map of quotients. Then there is a unique cohomomorphism $\underline{[e]} : \vec{p}'\mathscr{A}' \to \vec{p}\mathscr{A}$ such that the following diagram is commutative:

$$\begin{tikzcd}
\mathscr{A} \arrow[dr] & & & \mathscr{A}' \arrow[lll, "\underline{e}"'] \arrow[dl] \\
& X \arrow[r, "e"] \arrow[d, "p"] & X' \arrow[d, "p'"'] & \\
& Y \arrow[r, "{[e]}"] & Y' & \\
\vec{p}\mathscr{A} \arrow[ur] \arrow[uuu, "p^+"] & & & \vec{p}'\mathscr{A}' \arrow[ul] \arrow[lll, dashed, "\underline{[e]}"] \arrow[uuu, "(p')^+"']
\end{tikzcd}$$

Proof:

We can construct $\underline{[e]}$ as follows. Take $\underline{e} \circ (p')^{+} \in (p' \circ e)\text{-Cohom}(\vec{p}'\mathcal{A}'; \mathcal{A})$ and factor as $\underline{e} \circ (p')^{+} = e^{+} \circ (p')^{+} \circ \varepsilon$ where $\varepsilon \in \mathrm{Hom}(\vec{p}'\mathcal{A}'; \vec{p}'(\vec{e}\mathcal{A})) = \mathrm{Hom}(\vec{p}'\mathcal{A}'; \overrightarrow{[e]}(\vec{p}\mathcal{A}))$. This, in turn, defines an element $\underline{[e]} \in [e]\text{-Cohom}(\vec{p}'\mathcal{A}'; \vec{p}\mathcal{A})$ by $\underline{[e]} = [e]^{+} \circ \varepsilon$. Commutativity is obvious from the definition. Unicity follows from the stalkwise description; in fact, $\underline{[e]}$ must be a direct sum between the pushed forward stalks whose components are given by $\underline{e}$. We give the details in 1.5.5. ■

1.5.5 AN EXPLICIT DESCRIPTION OF $\underline{[e]}$

We describe $\underline{[e]}$ explicitly. We have three cases:

1). $[x', a', x'] \in Y' \backslash \mathcal{L}'$ an orbit and $[e][x, ax] = [x', a'x']$, i.e., $ex = x'$.

Then:
$$(\vec{p}\mathcal{A})|_{[x, ax]} = \mathcal{A}|_{x} \oplus \mathcal{A}|_{ax}$$
$$(\vec{p}'\mathcal{A}')|_{[x', a'x']} = \mathcal{A}'|_{x'} \oplus \mathcal{A}'|_{a'x'}$$

and $\underline{[e]}$ is given by $(\underline{e}|_{x' \to x}) \oplus (\underline{e}|_{a'x' \to ax})$ on the stalks.

2). $[x'] \in \mathcal{L}'$ and $[e][x, ax] = [x']$, i.e., $ex = x'$.

Then:
$$(\vec{p}\mathcal{A})|_{[x, ax]} = \mathcal{A}|_{x} \oplus \mathcal{A}|_{ax}$$
$$(\vec{p}'\mathcal{A}')|_{[x']} = \mathcal{A}'|_{x'}$$

and $\underline{[e]}$ is given by $(\underline{e}|_{x' \to x}) \oplus (\underline{e}|_{x' \to ax})$ on the stalks.

3). $[x'] \in \mathcal{L}'$ and $[x] \in \mathcal{L}$ with $[e][x] = [x']$, i.e., $ex = x'$.

Then:
$$\vec{p}\mathcal{A}|_{[x]} = \mathcal{A}|_{x}$$
$$\vec{p}'\mathcal{A}'|_{[x']} = \mathcal{A}'|_{x'}$$

and $\underline{[e]}$ is given by $(\underline{e}|_{x' \to x})$ on the stalks.

1.5.6 PROPOSITION

The reader may easily verify the following:

(1). $\underline{[e]}$ respects the composition; i.e. $\underline{[g \circ e]} = \underline{[g]} \circ \underline{[e]}$.

(2). Let the sheaves in 1.5.4 have some structure; i.e., let $\mathscr{A}$ and $\mathscr{A}'$ be sheaves of rings or modules over some constant sheaf of rings. Then, if we take the push forward structure on the pushed forward sheaves, $\underline{[e]}$ becomes a map in the stated category provided $\underline{e}$ was so.

(3). We have the following commutative diagram of rings:

$$\begin{tikzcd}
H^*(X', \mathscr{A}') \arrow[r, "\underline{e}"] & H^*(X, \mathscr{A}) \\
H^*(Y', \vec{p}'\mathscr{A}') \arrow[r, "\underline{[e]}"] \arrow[u, "\text{iso}", "\cong"'] & H^*(Y, \vec{p}\mathscr{A}) \arrow[u, "\text{iso}"', "\cong"]
\end{tikzcd}$$

provided $\underline{e}$ was a cohomomorphism of rings.

(4). Let $e : X \to X'$ be a map. Then e is always covered by a cohomomorphism $\underline{e} : Z_2 \to Z_2$ of constant sheaves.

1.5.7 LEMMA

Suppose $\mathscr{B}, \mathscr{B}'$ are sheaves on Y and $\gamma \in \mathrm{Hom}(\mathscr{B}', \mathscr{B})$. The there exists a unique a - cohomomorphism $\underline{a} : \overleftarrow{p}\mathscr{B}' \to \overleftarrow{p}\mathscr{B}$ such that the diagram

$$\begin{tikzcd} \overleftarrow{p}\mathscr{B}' \arrow[r, "\underline{a}"] \arrow[d, "p^*"'] & \overleftarrow{p}\mathscr{B} \arrow[d, "p^*"] \\ \mathscr{B}' \arrow[r, "\gamma"] & \mathscr{B} \end{tikzcd}$$

commutes. The construction respects composition.

Proof:

By 1.2,

$$\text{a-Cohom}(\overleftarrow{p}\mathscr{B}', \overleftarrow{p}\mathscr{B}) \cong \text{Hom}(\overleftarrow{p}\mathscr{B}', \overleftarrow{a}\,\overleftarrow{p}\mathscr{B}) = \text{Hom}(\overleftarrow{p}\mathscr{B}', \overleftarrow{p}\mathscr{B}).$$

By the functoriality of $\overleftarrow{p}$, we have the map

$$\overleftarrow{p} : \text{Hom}(\mathscr{B}', \mathscr{B}) \to \text{Hom}(\overleftarrow{p}\mathscr{B}', \overleftarrow{p}\mathscr{B}) .$$

The image of $\gamma \in \text{Hom}(\mathscr{B}', \mathscr{B})$ in $\text{a-Cohom}(\overleftarrow{p}\mathscr{B}', \overleftarrow{p}\mathscr{B})$ gives $\underline{a}$. ■

1.5.8 OBSERVATION

Notice that the stalkwise description of $\underline{a}$ is extremely simple; in fact, $(\underline{a}|_{X \to aX})$ is merely $\gamma|_{[x]}$ under the identification by p^*.

1.5.9 DEFINITION OF THE HOMOMORPHISMS α, σ AND ∇

Let (X, a) be a space with involution. Let $\mathscr{B}$ be a sheaf on Y. Taking γ to be the identity homomorphism, we can pull back γ by 1.5.7. Applying 1.5.4, we can push back the resulting cohomomorphism to obtain a homomorphism $\alpha : \overleftrightarrow{p}\mathscr{B} \to \overleftrightarrow{p}\mathscr{B}$. The stalkwise description of α is again very simple; from 1.5.5 and 1.5.8 it follows that α is the identity on $\overleftrightarrow{p}\mathscr{B}|_y$ for $y \in \ell$ and the natural "switch" for $y \in \tilde{\ell}$. Thus $\alpha \circ \alpha = \text{Id}$, i.e., α is a homomorphism of period two.

Define $\sigma = 1 + \alpha$. The stalkwise description of σ is as follows:

(1). $y \in \tilde{\ell}$, $\overset{\leftrightarrow}{p}\mathscr{B}|_y = \mathscr{B}|_y \oplus \mathscr{B}|_y$, and $\sigma(a, b) = (a+b, a+b)$ for $a, b \in \mathscr{B}|_y$.

(2). $y \in \ell$, $\overset{\leftrightarrow}{p}\mathscr{B}|_y = \mathscr{B}|_y$ and $\sigma = 0$.

There is an injection $\nabla : \mathscr{B} \to \overset{\leftrightarrow}{p}\mathscr{B}$ which is stalkwise given by the "diagonal" map $a \to (a, a)$. In fact, this was established in 1.2. It follows that we can imagine $\mathscr{B}$ as a subsheaf of $\overset{\leftrightarrow}{p}\mathscr{B}$.

We have the short exact sequence: $\operatorname{Ker}\sigma \rightarrowtail \overset{\leftrightarrow}{p}\mathscr{B} \twoheadrightarrow \operatorname{Im}\sigma$.

By the stalkwise description above, we can identify the terms as:

$$\operatorname{Ker}\sigma = \mathscr{B} \quad \text{and} \quad \operatorname{Im}\sigma = (\mathscr{B})_{\tilde{\ell}} \ .$$

All in all, we associated to a space with involution and a sheaf on the factor space a short exact sequence of the form:

$$S: \qquad \mathscr{B} \overset{\nabla}{\rightarrowtail} \overset{\leftrightarrow}{p}\mathscr{B} \overset{\sigma}{\twoheadrightarrow} (\mathscr{B})_{\tilde{\ell}} \ .$$

1.5.10

Let (X', a') be a space with involution. Let X be an a'-invariant paracompact subspace. We use the notation $a := a'|_X$, $p := p'|_X$, and $Y := p'(X)$. Then (X, a) is a space with involution, $Y = X/a$, and we have the commutative diagram

$$\begin{tikzcd} X \arrow[r, "i_X"] \arrow[d, "p"'] & X' \arrow[d, "p'"] \\ Y \arrow[r, "i_Y"] & Y' \end{tikzcd}$$

where i_X and i_Y are the natural subspace inclusions. Evidently, $L = L' \cap X$, and $\ell = \ell' \cap Y$.

Let $\mathscr{B}'$ be a sheaf on Y' and $\mathscr{B}$ denote $\mathscr{B}'|_Y$; i.e., $\mathscr{B} = \overleftarrow{i}_Y\mathscr{B}'$. Let S' denote the sequence as in 1.5.9 for the space (X', a') while S will be the analogous sequence for (X, a) . Since the pull back functor is exact, we have the commutative diagram $S' \xrightarrow[i_Y^*]{} \overleftarrow{i}_Y S'$ as follows:

$$\begin{tikzcd}
\mathscr{B}' \arrow[r, tail, "\nabla'"] \arrow[d, "(i_Y^*)'"'] & \overleftrightarrow{p}'\mathscr{B}' \arrow[r, two heads, "\sigma'"] \arrow[d, "(i_Y^*)''"] & (\mathscr{B}')_{\tilde{\jmath}} \arrow[d, "(i_Y^*)'''"] \\
\overleftarrow{i}_Y\mathscr{B}' \arrow[r, tail, "\overleftarrow{i}_Y(\nabla')"'] & \overleftarrow{i}_Y\overleftrightarrow{p}'\mathscr{B}' \arrow[r, two heads, "\overleftarrow{i}_Y(\sigma')"'] & \overleftarrow{i}_Y(\mathscr{B}')_{\tilde{\jmath}}
\end{tikzcd} .$$

1.5.11 LEMMA

$$\overleftarrow{i}_Y(\overleftrightarrow{p}'\mathscr{B}') \cong \overleftrightarrow{p}\mathscr{B} .$$

<u>Proof</u>:

Apply Lemma 1.5.4 to the special case when $e := i_X$, $\mathscr{A} := \overleftarrow{p}'\mathscr{B}'$, $\mathscr{A}' := \overleftarrow{i}_X(\overleftarrow{p}'\mathscr{B}')$, $\underline{e} := \underline{i}_X$. Since $[e] = i_Y$, that lemma implies the existence of a i_Y - cohomomorphism:

$$\underline{i}_Y : \overleftrightarrow{p}'\mathscr{B}' \to \vec{p}(\overleftarrow{i}_X(\overleftarrow{p}'\mathscr{B}')) = \vec{p}(\overleftarrow{p}\mathscr{B}) = \overleftrightarrow{p}\mathscr{B} .$$

The first equality holds because $i_X \circ p' = p \circ i_Y$.

$\underline{i}_Y$ is an isomorphism by the stalkwise description in 1.5.5; in fact, it is the identity on the stalks over Y and zero otherwise.

By standard adjuction, $\underline{i}_Y$ defines a sheaf homomorphism $\varepsilon : \overleftarrow{i}_Y(\overleftrightarrow{p}'\mathscr{B}') \to \overleftrightarrow{p}\mathscr{B}$ which is an isomorphism stalkwise. ■

1.5.12 LEMMA

$$\overleftarrow{i}_Y(\mathcal{B}')_{\tilde{\ell}} \cong (\mathcal{B})_{\tilde{\ell}} .$$

Proof:

$$\overleftarrow{i}_Y(\mathcal{B}')_{\tilde{\ell}}|_{\tilde{\ell}} = (\mathcal{B})_{\tilde{\ell}}|_{\tilde{\ell}} = \mathcal{B}'|_{\tilde{\ell}} .$$

Both sides give sheaves on Y with the property that the stalks are zero outside $\tilde{\ell}$, and they restrict on the same sheaf on $\tilde{\ell}$. Hence the two sheaves must agree. ■

1.5.13 LEMMA

$$\overleftarrow{i}_Y S' = S$$

Proof:

By the previous argument, we only have to verify that $\overleftarrow{i}_Y(\nabla') = \nabla$ and $\overleftarrow{i}_Y(\sigma') = \sigma$. But this immediately follows from the stalkwise description. ■

1.5.14 LEMMA

The diagram

$$\begin{tikzcd}
\overleftrightarrow{p}'\mathcal{B}' \arrow[d, "i_Y"'] & \overleftarrow{p}'\mathcal{B}' \arrow[l] \arrow[d, "i_X"] \\
\overleftrightarrow{p}\mathcal{B} & \overleftarrow{p}\mathcal{B} \arrow[l]
\end{tikzcd}$$

commutes.

Proof:

Trivial stalkwise verification gives the proof. ■

1.5.15 THEOREM

Let (X, a) be a space with involution. Let $\mathscr{B}$ be a sheaf on Y. Then there is a long exact sequence:

$$\begin{tikzcd}
 & H^*(X, \overleftarrow{p}\mathscr{B}) \arrow[dr, "\Delta"] & \\
H^*(Y, \mathscr{B}) \arrow[ur, "p^*"] & & H^*(Y, \ell, \mathscr{B}) \arrow[ll, "\mu"']
\end{tikzcd}$$

with connecting morphism μ.

The sequence is natural with respect to subspace inclusion; i.e., if X is an equivariant subspace of X', then

$$\begin{tikzcd}
 & H^*(X, \overleftarrow{p}(\mathscr{B}'|_X)) \arrow[dd, "\Delta"] & H^*(X', \overleftarrow{p}'\mathscr{B}') \arrow[l, "\underline{i}_X"'] \arrow[dd, "\Delta'"'] & \\
H^*(Y, \mathscr{B}'|_Y) \arrow[ur, "p^*"] & & & H^*(Y', \mathscr{B}') \arrow[lll, "(\underline{i}_Y)'"'] \arrow[ul, "(p')^*"'] \\
 & H^*(Y, \ell, \mathscr{B}'|_Y) \arrow[ul, "\mu"] & H^*(Y', \ell', \mathscr{B}') \arrow[l, "(\underline{i}_Y)''"'] \arrow[ur, "\mu'"'] &
\end{tikzcd}$$

commutes.

<u>Proof:</u>

The existence of the exact sequence follows from 1.5.9 and 1.5.6 (3). Indeed, our sequence is just the long exact sequence associated to the short exact sequence S in 1.5.9. The Vietoris - Begle isomorphism (see 1.2) and the commutativity of the diagram

$$\begin{tikzcd} & \overleftrightarrow{p}\mathscr{B} \arrow[d, "p^+"] \\ \mathscr{B} \arrow[ur, "\nabla"] \arrow[r, "p^*"'] & \overleftarrow{p}\mathscr{B} \end{tikzcd}$$

allows us to replace the $H^*(Y, \overleftrightarrow{p}\mathscr{B})$ term. Also, since ℓ is closed, we have that $H^*(Y, (\mathscr{B})_{\tilde{\ell}}) = H^*(Y, \ell, \mathscr{B})$. Equivariance now follows from statements 1.5.10 thru 1.5.14. ■

1.5.16 LEMMA

For the case $\mathscr{B} = \mathbb{Z}_2$, the constant sheaf, we have naturality for *all* equivariant maps $e : X \to X'$ (X is not necessarily a subspace of X').

Proof:

This easily follows from the fact that every map can be covered by a cohomomorphism of the constant sheaves. ■

1.5.17

We quickly review the construction of relative cohomology sequences in the case of a closed subspace.

Let X be a space and F a closed subspace. Then a short exact sequence of sheaves on X of the following form can be associated to any sheaf $\mathscr{A}$ on X :

$$R_F : \qquad (\mathscr{A})_{\tilde{F}} \overset{\iota_F}{\rightarrowtail} \mathscr{A} \overset{\rho_F}{\twoheadrightarrow} (\mathscr{A})_F$$

The corresponding long exact sequence gives the relative cohomology sequence of the pair (X, F) (which can be obtained for any subspace via a more general definition). Just as the construction of the sequence S, R_F has the naturality property; i.e., if $X \subset X'$ and $F' \subset X'$ with X paracompact, then $\overleftarrow{i}_X R'_{F'}$ gives $R_{F' \cap X}$ on X. More generally, if $(X, F) \subset (X', F')$, then i_X gives a cohomomorphism $\underline{i}_X$ of $R'_{F'}$ into R_F, which is the composition of the cohomomorphism $i_X^* : R'_{F'} \to R_{F' \cap X}$ and the homomorphism $R_{F' \cap X} \to R_X$.

1.5.18

Let (X, a) be a space with involution. Fix a sheaf $\mathscr{B}$ on Y. Let $\mathfrak{n}$ be any closed subspace in Y and $N := p^{-1}(\mathfrak{n})$. Let us associate the sheaf $(\mathscr{B})_{\tilde{\mathfrak{n}}}$ to $\mathfrak{n}$. If we apply Theorem 1.5.15 to the sheaf $(\mathscr{B})_{\tilde{\mathfrak{n}}}$, we get the following exact sequence.

1.5.19 THEOREM

Under the conditions above, there is a long exact sequence:

$$S_N : \quad \begin{tikzcd} & H^*(X, N, \overleftarrow{p}\mathscr{B}) \arrow[dr, "\Delta_N"] & \\ H^*(Y, \mathfrak{n}, \mathscr{B}) \arrow[ur, "p_N^*"] & & H^*(Y, \ell \cup \mathfrak{n}, \mathscr{B}) \arrow[ll, "\mu_N"] \end{tikzcd} .$$

This sequence is natural with respect to subspace inclusions. If $\mathscr{B} = \mathbb{Z}_2$, then the sequence is natural with respect to general equivariant maps of pairs. In fact, $\overleftarrow{p}(\mathscr{B})_{\tilde{\mathfrak{n}}} = (\overleftarrow{p}\mathscr{B})_{\tilde{N}}$ and $((\mathscr{B})_{\tilde{\mathfrak{n}}})_{\tilde{\ell}} = (\mathscr{B})_{\widetilde{\mathfrak{n} \cap \ell}}$, just as in 1.5.12. Naturality follows from the fact that both S and R are natural.

1.6 A COMMUTATIVE DIAGRAM

In this section, we state and prove a technical lemma which will play an important role in later computations.

1.6.1 LEMMA

We have the following commutative diagram:

$$
\begin{tikzcd}
 & & & H^n(Y, \mathscr{B}) \arrow[dl, "p_0^*"'] \arrow[drrr, "i_\ell"] \arrow[d, phantom, "(1)"] & & & & \\
H^n(X, L, \overleftarrow{p}\mathscr{B}) \arrow[rr] \arrow[dr, "\Delta_L"'] & \arrow[d, phantom, "(4)"] & H^n(X, \overleftarrow{p}\mathscr{B}) \arrow[rr, "i_L"'] \arrow[dl, "\Delta_0"] & \arrow[d, phantom, "(5)"] & H^n(L, \overleftarrow{p}\mathscr{B}|_L) \arrow[rr, phantom, "\equiv"] \arrow[d, "\delta_\ell"'] & \arrow[d, phantom, "(2)"] & H^n(\ell, \mathscr{B}|_\ell) \arrow[dr, "\delta_L"] & \\
 & H^n(Y, \ell, \mathscr{B}) \arrow[rrr, "\mu_L"'] \arrow[dr, "\mu_0"'] & \arrow[d, phantom, "(3)"] & & H^{n+1}(Y, \ell, \mathscr{B}) \arrow[rrr, "p_L^*"'] \arrow[dll, "p_\ell"] & & & H^{n+1}(X, L, \overleftarrow{p}\mathscr{B}) \\
 & & H^{n+1}(Y, \mathscr{B}) & & & & &
\end{tikzcd}
$$

where the groups and maps appearing above have been defined as in the previous chapter. Notice that the maps in the diagram are either maps in a long exact sequence of pairs, or, morphisms from the Smith sequences associated to $N = \varnothing$ or $N = L$.

<u>Proof</u>:

(a). Triangles (1) and (2) are obviously commutative since long exact sequences of pairs are natural with respect to cohomomorphism covering a map of pairs.

(b). Commutativity for triangles (3) and (4) follows from the naturality of the Smith sequence.

(c). The nontrivial fact is the commutativity of the central part, (5). To prove the commutativity of this square, we use the Godement representation we developed in section 1.4. We will obtain commutativity on chain level for the representatives in $A/\underline{A}$. For further explanation see the diagram below. In this diagram,

(1). Squares (1) - (4) are obviously commutative on A - level since they are induced by cohomomorphisms that commute on sheaf level.

(2). Squares (5) - (6) commute modulo terms in $\underline{A}$ since on the quotient level they give the induced map of the canonical flabby resolution which is supposed to be a chain map.

(3). Dashed arrows exist on cohomology level and thus (by definition) make the diagram "commutative" in the appropriate sense.

Now our proof goes as follows. Take $c \in H^n(X, \overleftarrow{p}\mathscr{B})$. Take a representative cochain $f(z_0, z_1, ..., z_n) \in A(Y, \overleftrightarrow{p}\mathscr{B}, n)$ for $[c]$; in doing so we used the Vietoris - Begle theorem to give us the isomorphism $H^n(X, \overleftarrow{p}\mathscr{B}) \cong H^n(Y, \overleftrightarrow{p}\mathscr{B})$. Decompose f as $f = f_1 + f_2$ where

$$f_1(z_0, z_1, ..., z_n) := \begin{cases} f(z_0, z_1, ..., z_n) & \text{if } z_n \in \ell \\ 0 & \text{otherwise} \end{cases}$$

$$f_2(z_0, z_1, ..., z_n) := \begin{cases} f(z_0, z_1, ..., z_n) & \text{if } z_n \notin \ell \\ 0 & \text{otherwise} \end{cases}$$

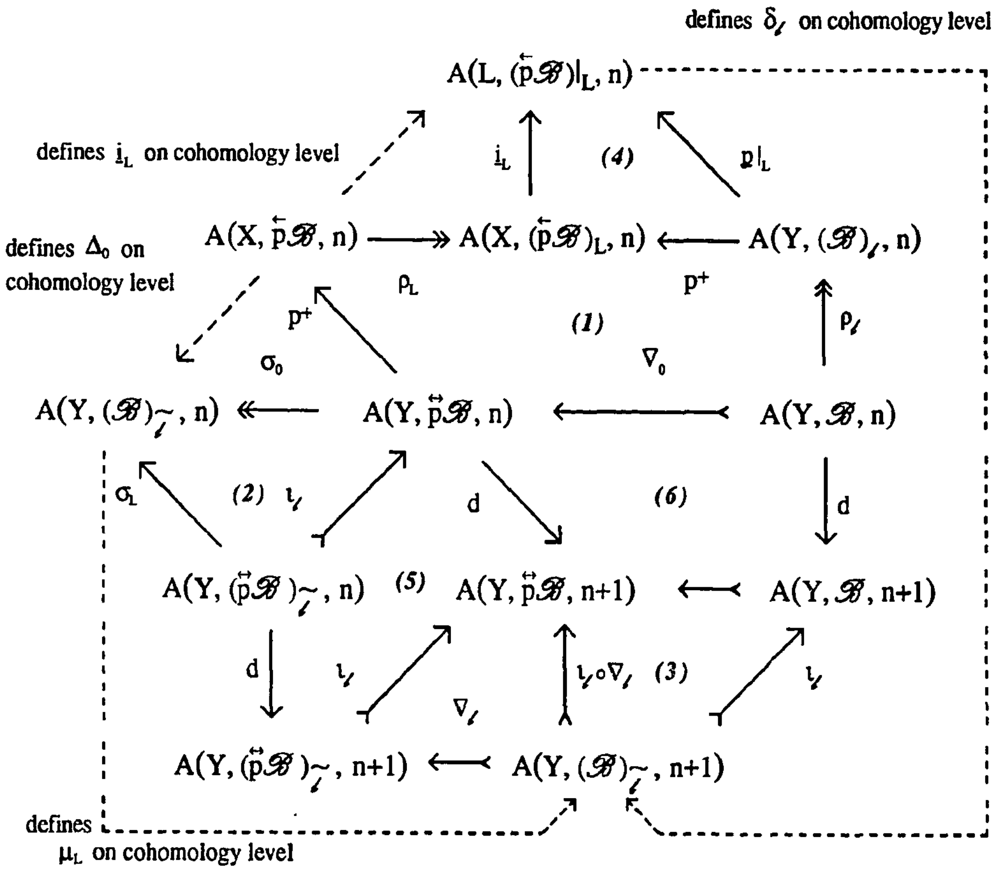
Then, in fact,

$$f_1 \in A(Y, \mathscr{B}, n)$$

and $$f_2 \in A(Y, (\overleftrightarrow{p}\mathscr{B})_{\tilde{\ell}}, n) .$$

Also, $df_1 + df_2 = df \in \underline{A}(Y, \overleftrightarrow{p}\mathscr{S}, n+1)$ since f is a cocycle. Now, by the commutativity of the diagram, easy diagram chasing shows that $df_1 + df_2$ can be viewed as $(\iota_\ell \circ \nabla_\ell)g$ where $g \in A(Y, (\overleftrightarrow{p}\mathscr{S})_{\tilde{\ell}}, n+1)$ is a representative of the commutator $(\mu_L \circ \Delta_0 \circ \delta_\ell \circ \underline{i}_L)(c)$. Since $\iota_\ell \circ \nabla_\ell$ is an injection of sheaves, this implies that $g \in \underline{A}(Y, (\overleftrightarrow{p}\mathscr{S})_{\tilde{\ell}}, n+1)$, i.e., the commutator is 0 . ■

1.6.2 DIAGRAM



1.7 STEENROD SQUARES: THE GENERAL CONSTRUCTION

In this section we discuss the existence and unicity of Steenrod squares in sheaf theoretical context. (An explicit construction more suitable for further computations will carried out in 1.8.) Throughout this part of the text we conform to the notation conventions of [Bredon 1].

1.7.1 OBSERVATION

In this section and the following, we will be working in the category of sheaves of $\mathbb{Z}_2$ - algebras. This supposition greatly simplifies many of the proofs that otherwise can be carried out in greater generality. Let $\mathscr{A}$ be a sheaf on a paracompact Hausdorff space X. The standard canonical flabby resolution $\{\mathscr{C}^*(X, \mathscr{A}), d\}$ inherits the $\mathbb{Z}_2$ - algebra structure and becomes a resolution in our category. In fact:

(a). $\{\mathscr{C}^*(X, \mathscr{A}), d\}$ is an injective resolution. Notice that this statement fails to be true in more general categories, for example the category of sheaves of abelian groups. Let $\{\mathscr{C}^*(X, \mathscr{A}) \otimes \mathscr{C}^*(X, \mathscr{A}), d_\otimes\}$ be the tensor product of resolutions. Here $d_\otimes$ on $\mathscr{C}^*(X, \mathscr{A}) \otimes \mathscr{C}^*(X, \mathscr{A})$ is defined by $d_\otimes(c_1 \otimes c_2) = dc_1 \otimes c_2 + c_1 \otimes dc_2$. We have:

(b). $\{\mathscr{C}^*(X, \mathscr{A}) \otimes \mathscr{C}^*(X, \mathscr{A}), d_\otimes\}$ is a resolution of $\mathscr{A} \otimes \mathscr{A}$. While this obviously follows from the fact that short exact sequences split over fields, notice that $\mathscr{C}^*(X, \mathscr{A})$ splits pointwise anyway. Hence (b) is not specific to our category.

1.7.2 DEFINITION

Let $\nu : \mathscr{C}^*(X, \mathscr{A}) \otimes \mathscr{C}^*(X, \mathscr{A}) \to \mathscr{C}^*(X, \mathscr{A}) \otimes \mathscr{C}^*(X, \mathscr{A})$ be the natural homomorphism given by the "switch map" stalkwise; i.e., $\nu(c_1 \otimes c_2) := (c_2 \otimes c_1)$. Let $\tau := \mathrm{Id} + \nu$. Then τ is a map of differential sheaves: $\tau \circ d_\otimes = d_\otimes \circ \tau$. Also, $\tau \circ \tau = 0$.

1.7.3 CONSTRUCTION

Suppose $\chi : \mathscr{A} \otimes \mathscr{A} \to \mathscr{A}$ is the ring structure on $\mathscr{A}$; this is evidently a map in our category. We then use χ to define a system of maps $\langle \chi_k \mid k = 0, 1, \ldots \rangle$, with $\chi_k := \mathscr{C}^*(X, \mathscr{A}) \otimes \mathscr{C}^*(X, \mathscr{A}) \to \mathscr{C}^*(X, \mathscr{A})$ being a map of graded sheaves of degree $(-k)$. With the exception of χ_0, χ_k does *not* respect the differentiation; in fact, χ_k is a sort of higher order commutator. χ_0 will extend χ on augmentation level. We proceed by induction as follows.

(1). $\chi : \mathscr{A} \otimes \mathscr{A} \to \mathscr{A}$ extends to a chain map $\chi_0 : \mathscr{C}^*(X, \mathscr{A}) \otimes \mathscr{C}^*(X, \mathscr{A}) \to \mathscr{C}^*(X, \mathscr{A})$ by 1.7.1 (a), 1.7.1 (b), and 1.2.8.

(2). The map $\chi_0 \circ \tau$ extends $\chi \circ (\tau|_{\mathscr{A} \otimes \mathscr{A}}) = 0$. By 1.2.8 again, $\chi_0 \circ \tau$ is chain homotopic to the zero map. Consequently, there exists a homotopy

$\chi_1 : \mathscr{C}^*(X, \mathscr{A}) \otimes \mathscr{C}^*(X, \mathscr{A}) \to \mathscr{C}^*(X, \mathscr{A})$ such that

$$(*)_1 \qquad \chi_0 \circ \tau = \chi_1 \circ d_{\otimes} + d \circ \chi_1 .$$

By construction, χ_1 can be chosen with the property that

$$\chi_1|_{[\mathscr{C}^*(X, \mathscr{A}) \otimes \mathscr{C}^*(X, \mathscr{A})]_1} = 0 .$$

(3). Applying τ on both sides in $(*)$, we get that $\chi_1 \circ \tau$ is a chain map that (by choice) extends the zero map. By repeating the argument in (2) we can construct a homotopy

$\chi_2 : \mathscr{C}^*(X, \mathscr{A}) \otimes \mathscr{C}^*(X, \mathscr{A}) \to \mathscr{C}^*(X, \mathscr{A})$ such that

$$(*)_2 \qquad \chi_1 \circ \tau = \chi_2 \circ d_{\otimes} + d \circ \chi_2$$

and $\chi_2|_{[\mathscr{C}^*(X, \mathscr{A}) \otimes \mathscr{C}^*(X, \mathscr{A})]_2} = 0 .$

(4). Repeating the process, we construct homotopies

$\chi_{k+1} : \mathscr{C}^*(X, \mathscr{A}) \otimes \mathscr{C}^*(X, \mathscr{A}) \to \mathscr{C}^*(X, \mathscr{A})$ of degree $-(k+1)$ such that

$$(*)_k \qquad \chi_k \circ \tau = \chi_{k+1} \circ d_{\otimes} + d \circ \chi_{k+1}$$

$$\text{and} \qquad \chi_{k+1}|_{[\mathscr{C}^*(X,\mathscr{A}) \otimes \mathscr{C}^*(X,\mathscr{A})]_{k+1}} = 0 .$$

(5). Notice that there have been numerous choices involved in the process. For now, we fix a particular system of $\langle \chi_k \mid k = 0, 1, \ldots \rangle$.

1.7.4 DEFINITION

Let $\theta : \mathscr{C}^*(X,\mathscr{A}) \to \mathscr{C}^*(X, \mathscr{A}) \otimes \mathscr{C}^*(X,\mathscr{A})$ be the diagonal inclusion given by $\theta|_X : c \to c \otimes c$ stalkwise. We then have the induced map

$\theta : \mathscr{C}^*_X(X,\mathscr{A}) \to \mathscr{C}^*_X(X, \mathscr{A}) \otimes \mathscr{C}^*_X(X,\mathscr{A})$, which we compose with χ_k to obtain the system $\left\langle \mathrm{St}_k : \mathscr{C}^i_X(X, \mathscr{A}) \to \mathscr{C}^{2i-k}_X(X, \mathscr{A}) \mid k = 0, 1, \ldots \right\rangle$. Since neither χ_k nor θ is a chain map, it is not at all obvious that St_k descends to yield anything on cohomology level. In fact, we shall verify the following statements.

(1). If $c \in \mathscr{C}^*_X(X,\mathscr{A})$ is a cocycle, then $\mathrm{St}_k(c)$ is a cocycle.

(2). If $c \in \mathscr{C}^*_X(X,\mathscr{A})$ is a coboundary, then $\mathrm{St}_k(c)$ is a coboundary.

(3). If c_1 and c_2 are cocycles in $\mathscr{C}^*_X(X,\mathscr{A})$, then $\mathrm{St}_k(c_1 + c_2) - \mathrm{St}_k(c_1) - \mathrm{St}_k(c_1)$ is a coboundary.

Granted the results above, we have the following theorem.

1.7.5 THEOREM [STEENROD - BREDON]

Let $\mathscr{A}$ be a sheaf of $\mathbb{Z}_2$ - algebras over X. Then we can define a sequence of homomorphisms $\left\langle St_k : H^i(X, \mathscr{A}) \to H^{2i-k}(X, \mathscr{A}) \mid k = 0, 1, \dots \right\rangle$ with St_0 being the cup product induced by χ. ∎

1.7.6 LEMMA

Let $c \in \mathscr{C}_X^*(X, \mathscr{A})$ be a cocycle. Then $(d \circ \chi_k \circ \theta)c = 0$. As a consequence, 1.7.4 (1) holds.

Proof:

$$(d \circ \chi_k \circ \theta)c := (d \circ \chi_k)(c \otimes c) = \chi_k(dc \otimes c) + \chi_k(c \otimes dc) = 0 . \quad ∎$$

1.7.7 LEMMA

Let $c \in \mathscr{C}_X^*(X, \mathscr{A})$. Then $(\chi_k \circ \theta \circ d)(c) = (d \circ \chi_k)(c \otimes dc) + (d \circ \chi_{k-1})(c \otimes c)$. As a consequence, 1.7.4 (2) holds.

Proof:

$$(d \circ \chi_k)(c \otimes dc) = \chi_{k-1}(c \otimes dc + dc \otimes c) + \chi_k(dc \otimes dc) ;$$

$$(d \circ \chi_{k-1})(c \otimes c) = \chi_{k-1}(dc \otimes c + c \otimes dc) . \quad ∎$$

1.7.8 LEMMA

Let c_1 and c_2 be cocycles in $\mathscr{C}_X^*(X, \mathscr{A})$.

Then $\chi_k((c_1 + c_2) \otimes (c_1 + c_2)) + \chi_k(c_1 \otimes c_1) + \chi_k(c_2 \otimes c_2) = d\chi_{k+1}(c_1 \otimes c_2)$.

As a consequence, 1.7.4 (3) holds.

Proof:

$$
\begin{aligned}
&\chi_k((c_1+c_2)\otimes(c_1+c_2)) + \chi_k(c_1\otimes c_1) + \chi_k(c_2\otimes c_2)\\
= \;& \chi_k(c_1\otimes c_2 \;+\; c_2\otimes c_1)\\
= \;& (\chi_k\circ\tau)(c_1\otimes c_2)\\
= \;& (d\circ\chi_{k+1})(c_1\otimes c_2). \quad\blacksquare
\end{aligned}
$$

1.7.9 OBSERVATION

Let $\mathscr{M}^*(X, \mathscr{A})$ be an arbitrary acyclic resolution of $\mathscr{A}$. Let $\mathscr{N}^*(X, \mathscr{A})$ be an injective resolution. Let $\langle \chi_k \mid k = 0, 1, \dots \rangle$ be an arbitrary system of sheaf maps with $\chi_k : \mathscr{M}^*(X, \mathscr{A}) \otimes \mathscr{M}^*(X, \mathscr{A}) \to \mathscr{N}^*(X, \mathscr{A})$ of degree (-k) with the property as in 1.7.3 4). Then, by the arguments above, we can construct a system of homomorphisms $\langle St_k \mid k = 0, 1, \dots \rangle$ on cohomology level. We may ask whether this system is the same as we constructed earlier in terms of serrations and an arbitrarily chosen homotopy system; in particular, whether the homomorphisms in 1.7.5 are independent of the choices made in 1.7.3. The answer is yes, it is. That is, the system $\langle St_k \mid k = 0, 1, \dots \rangle$ in 1.7.5 is a canonically defined system. To verify this, we essentially have to check the validity of the following three statements.

(1). For a fixed system of resolutions $\mathscr{M}^*(X, \mathscr{A})$ and $\mathscr{N}^*(X, \mathscr{A})$, any system with the property $(*)_k$ gives rise to the same system of maps $\langle St_k \mid k = 0, 1, \dots \rangle$ on cohomology level. Thus we may talk of the operation defined by $\mathscr{M}^*(X, \mathscr{A})$ and $\mathscr{N}^*(X, \mathscr{A})$.

(2). If $\widetilde{\mathscr{N}}^*(X, \mathscr{A})$ is another injective resolution, then $(\mathscr{M}^*(X, \mathscr{A}), \mathscr{N}^*(X, \mathscr{A}))$ and $(\mathscr{M}^*(X, \mathscr{A}), \widetilde{\mathscr{N}}^*(X, \mathscr{A}))$ define the same system of maps on cohomology level. Consequently, we can talk about the operation defined by $\mathscr{M}^*(X, \mathscr{A})$.

(3). If $\mathcal{M}^*(X, \mathcal{A})$ and $\widetilde{\mathcal{M}}^*(X, \mathcal{A})$ are resolutions, then $(\mathcal{M}^*(X, \mathcal{A}), \mathcal{N}^*(X, \mathcal{A}))$ and $(\widetilde{\mathcal{M}}^*(X, \mathcal{A}), \mathcal{N}^*(X, \mathcal{A}))$ define the same system of maps on cohomology level; i.e., the definition of $\langle St_k \mid k = 0, 1, \dots \rangle$ is "canonical".

1.7.10 LEMMA

Let $\mathcal{M}^*(X, \mathcal{A})$ and $\mathcal{N}^*(X, \mathcal{A})$ be as in 1.7.8. Let $\langle \chi_k \mid k = 0, 1, \dots \rangle$ and $\langle \chi'_k \mid k = 0, 1, \dots \rangle$ be two systems with

(a). $\chi_0|_{\mathcal{A}\otimes\mathcal{A}} = \chi = \chi'_0\,|_{\mathcal{A}\otimes\mathcal{A}}$

(b). $\chi_k|_{[\mathcal{M}^*(X,\mathcal{A})\otimes\mathcal{M}^*(X,\mathcal{A})]_k} = \chi'_k\,|_{[\mathcal{M}^*(X,\mathcal{A})\otimes\mathcal{M}^*(X,\mathcal{A})]_k} = 0$ and

(c). Property $(*)_k$ in 1.7.3 (4) holds for χ_k and χ'_k .

Then we have that there is a system of chain homotopies of degree (-k) that is of the form $\{ \gamma_k : \mathcal{M}^*(X, \mathcal{A}) \otimes \mathcal{M}^*(X, \mathcal{A}) \to \mathcal{N}^*(X, \mathcal{A}) \mid k = 0, 1, \dots \}$, and such that

$$\chi_k + \chi'_k + \gamma_k \circ \tau = \gamma_{k+1} \circ d_\otimes + d \circ \gamma_{k+1} .$$

Notice that τ is zero on the diagonal; i.e., $\tau|_{\mathrm{Im}\theta} = 0$. This implies that $\langle St_k \mid k = 0, 1, \dots \rangle$ and $\langle St'_k \mid k = 0, 1, \dots \rangle$ induce the same system of maps in cohomology; and consequently, that 1.7.9 (1) holds.

<u>Proof</u>:

χ_0 and χ'_0 are both chain maps extending $\chi : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$. By 1.2.8, there must exist a chain homotopy γ_1 with

$$\chi_0 + \chi'_0 = \gamma_1 \circ d_\otimes + d \circ \gamma_1 .$$

We can also suppose that $\gamma_1|[\mathcal{M}^*(X,\mathcal{A})\otimes\mathcal{M}^*(X,\mathcal{A})]_1 = 0$. From now on, our construction copies the argument in 1.7.3. Applying τ to both sides of the above equation yields:

$$d\circ(\gamma_1\circ\tau) + (\gamma_1\circ\tau)\circ d_\otimes = \chi_0\circ\tau + \chi_0'\circ\tau$$
$$= \chi_1\circ d + d\circ\chi_1 + \chi_1'\circ d + d\circ\chi_1' ,$$

or $\quad (\chi_1+\chi_1' + \gamma_1\circ\tau)\circ d + d\circ(\chi_1+\chi_1' + \gamma_1\circ\tau) = 0$.

Consequently, $(\chi_1+\chi_1' + \gamma_1\circ\tau)$ is a chain map extending the zero map on $[\mathcal{M}^*(X,\mathcal{A})\otimes\mathcal{M}^*(X,\mathcal{A})]_1$. By 1.2.8 again, there is a map γ_2 with $\gamma_2|[\mathcal{M}^*(X,\mathcal{A})\otimes\mathcal{M}^*(X,\mathcal{A})]_2 = 0$ such that

$$\chi_1+\chi_1' = \gamma_2\circ d_\otimes + d\circ\gamma_2 .$$

Inductively, we can construct the system $\left\langle \gamma_k \mid k=0,1,\dots \right\rangle$. ■

1.7.11 LEMMA

Statement (2) in 1.7.9 is true; that is, if $\widetilde{\mathcal{N}}^*(X,\mathcal{A})$ is another injective resolution, then $(\mathcal{M}^*(X,\mathcal{A}), \mathcal{N}^*(X,\mathcal{A}))$ and $(\mathcal{M}^*(X,\mathcal{A}), \widetilde{\mathcal{N}}^*(X,\mathcal{A}))$ define the same system of maps on cohomology level.

Proof:

By 1.2.8 there is a chain map $\gamma : \mathcal{N}^*(X,\mathcal{A}) \to \widetilde{\mathcal{N}}^*(X,\mathcal{A})$ which extends the identity on $\mathcal{A}$. Obviously, γ induces an isomorphism on cohomology level. Let $\left\langle \chi_k \mid k=0,1,\dots \right\rangle$ be a system as in 1.7.9 for the pair $(\mathcal{M}^*(X,\mathcal{A}), \mathcal{N}^*(X,\mathcal{A}))$. Then $\left\langle \gamma\circ\chi_k \mid k=0,1,\dots \right\rangle$ gives us a system for the pair $(\mathcal{M}^*(X,\mathcal{A}), \widetilde{\mathcal{N}}^*(X,\mathcal{A}))$ which satisfies the requirements as stated in 1.7.9. Thus by 1.7.10, the statement is proved. ■

1.7.12 OBSERVATION

Next, we will prove the naturality of the construction outlined in 1.7.9. We shall obtain statement (3) in 1.7.9 as a consequence.

Suppose $\mathscr{A}$ and $\mathscr{A}'$ are two sheaves on the topological spaces X and X' respectively. Let $\mathscr{M}^*(X, \mathscr{A})$ and $\mathscr{M}^*(X', \mathscr{A}')$ be acyclic resolutions. Suppose the map $k : X \to X'$ is covered by a cohomomorphism of sheaves $\underline{k} : \mathscr{A} \to \mathscr{A}'$. Since we work in the Z_2 - algebra category, $\underline{k}$ is required to make the following diagram commutative.

$$\begin{tikzcd} \mathscr{A} \otimes \mathscr{A} \arrow[r, "\chi"] \arrow[d, "\underline{k} \otimes \underline{k}"'] & \mathscr{A} \arrow[d, "\underline{k}"] \\ \mathscr{A}' \otimes \mathscr{A}' \arrow[r, "\chi'"'] & \mathscr{A}' \end{tikzcd} .$$

Suppose $\underline{k}$ has an extension $\underline{k} : \mathscr{M}^*(X, \mathscr{A}) \to \mathscr{M}^*(X', \mathscr{A}')$. Let $\mathscr{N}^*(X, \mathscr{A})$ and $\mathscr{N}^*(X', \mathscr{A}')$ be injective resolutions. Suppose once again that $\underline{k}$ has an extension $\underline{k} : \mathscr{N}^*(X, \mathscr{A}) \to \mathscr{N}^*(X', \mathscr{A}')$. Notice that for the choice $\mathscr{M}^*(X, \mathscr{A}) = \mathscr{N}^*(X, \mathscr{A}) = \mathscr{C}^*(X, \mathscr{A})$, $\mathscr{M}^*(X', \mathscr{A}') = \mathscr{N}^*(X', \mathscr{A}') = \mathscr{C}^*(X', \mathscr{A}')$ always satisfies the conditions above.

Form the operators $\langle St_k \mid k = 0, 1, \ldots \rangle$ and $\langle St'_k \mid k = 0, 1, \ldots \rangle$ determined by the resolutions above. Then we have the following statement.

1.7.13 LEMMA

The following diagram commutes; i.e., the construction of the Steenrod squares is natural.

$$\begin{tikzcd}
H^*(X, \mathscr{A}) \arrow[r, "St_k"] \arrow[d, "\underline{k}"'] & H^*(X, \mathscr{A}) \arrow[d, "\underline{k}"] \\
H^*(X', \mathscr{A}') \arrow[r, "St_k'"'] & H^*(X', \mathscr{A}')
\end{tikzcd} .$$

<u>Proof</u>:

See **[Bredon 1]** ■

This lemma has two important consequences.

1.7.14 CONSEQUENCE

Statement (3) in 1.7.9 is true; if $\mathscr{M}^*(X, \mathscr{A})$ and $\widetilde{\mathscr{M}}^*(X, \mathscr{A})$ are resolutions, then $(\mathscr{M}^*(X, \mathscr{A}), \mathscr{N}^*(X, \mathscr{A}))$ and $(\widetilde{\mathscr{M}}^*(X, \mathscr{A}), \mathscr{N}^*(X, \mathscr{A}))$ define the same system of maps on cohomology level; that is, the definition of $\langle St_k \mid k = 0, 1, \dots \rangle$ is "canonical".

<u>Proof</u>:

Let $\mathscr{M}^*(X, \mathscr{A})$ be an arbitrary acyclic resolution. Since the resolution $\mathscr{C}^*(X, \mathscr{A})$ is injective, the identity cohomomorphism $\underline{k} = \mathrm{Id}$ extends to $\underline{k}$: $\mathscr{M}^*(X, \mathscr{A}) \to \mathscr{C}^*(X, \mathscr{A})$. Moreover, $\underline{k}$ induces the identity on cohomology level. Now the commutativity of the diagram in 1.7.13 in this special case is precisely the statement that $\mathscr{M}^*(X, \mathscr{A})$ defines the same cohomology operation as $\mathscr{C}^*(X, \mathscr{A})$; or, that the operation is independent of the resolution $\mathscr{M}^*(X, \mathscr{A})$.

■

1.7.15 CONSEQUENCE

Let (X, a), Y, $p : X \to Y$ be defined as in 1.5.1. Let $\mathscr{A}$ be a sheaf of $\mathbb{Z}_2$-algebras on X. Equip $\vec{p}\mathscr{A}$ with the pushed-forward $\mathbb{Z}_2$-algebra structure. Then the natural cohomomorphism $p^+ : \vec{p}\mathscr{A} \to \mathscr{A}$ becomes a morphism in our category. The choices of $\mathscr{C}^*(X, \mathscr{A})$ and $\mathscr{C}^*(Y, \vec{p}\mathscr{A})$ evidently make the extension of p^+ possible; now the commutativity in 1.7.13 implies that the diagram

$$\begin{tikzcd} H^*(X, \mathscr{A}) \arrow[r, "St_k"] & H^*(X, \mathscr{A}) \\ H^*(Y, \vec{p}\mathscr{A}) \arrow[u, "p^+"', "\cong"] \arrow[r, "St_k"] & H^*(X, \vec{p}\mathscr{A}) \arrow[u, "p^+"', "\cong"] \end{tikzcd}$$

is commutative. I.e., the Vietoris - Begle isomorphism respects the cohomology operations defined in 1.7.5. This fact can be regarded as an extension of the statement that p^+ was a ring isomorphism.

1.7.16 DEFINITION

The system $\langle St_k \mid k = 0, 1, \dots \rangle$ constructed in 1.7.5 is called *the system of lower Steenrod operations*. Define $Sq^k := St_{n-k}$ on $H^n(X, \mathscr{A})$, $k = 0, 1, \dots$; i.e., re-index the set $\langle St_k \mid k = 0, 1, \dots \rangle$. The resulting system of maps $\langle Sq^k \mid k = 0, 1, \dots \rangle$ is called *the system of Steenrod squares*.

1.7.17 REMARK

If $\mathscr{A}$ is the constant sheaf $\mathbb{Z}_2$, then the system of Steenrod operations defined above via sheaf theory satisfies the Steenrod axioms on the category of finite simplicial complexes (see [Steenrod - Epstein]). Since these axioms uniquely characterize the

Steenrod square operations, the sheaf theoretical definition properly extends the original construction. That is, they coincide on finite simplicial complexes or, more generally, on CW - complexes.

1.8 A CANONICAL CONSTRUCTION

In this section we give an explicit homotopy system which induces the Steenrod square operations. This particular system possesses a certain naturality property which will play a crucial role in our further computations.

1.8.1

Suppose $\mathscr{A}$ is a subsheaf of $\mathscr{B}$ in the category of sheaves of $\mathbb{Z}_2$ - algebras over the space X . Since the canonical flabby resolution functor is exact, we have the imbedding $\mathscr{C}^*(X, \mathscr{A}) \rightarrowtail \mathscr{C}^*(X, \mathscr{B})$ of differential graded sheaves. The fact that we work with coefficients over a field yields the further imbedding:

$$\mathscr{C}^*(X, \mathscr{A}) \otimes \mathscr{C}^*(X, \mathscr{A}) \rightarrowtail \mathscr{C}^*(X, \mathscr{B}) \otimes \mathscr{C}^*(X, \mathscr{B})$$

of differential sheaves.

1.8.2

As we have seen in 1.7.5, the multiplication gives rise to a system of maps $\langle \chi_k \mid k = 0, 1, \ldots \rangle$ with certain symmetry properties where $\chi_k : \mathscr{C}^*(X, \mathscr{B}) \otimes \mathscr{C}^*(X, \mathscr{B}) \to \mathscr{C}^*(X, \mathscr{B})$ is a map of degree (-k) and χ_0 extends the multiplication $\chi : \mathscr{B} \otimes \mathscr{B} \to \mathscr{B}$. We have also seen that we may have more than one option in choosing the system $\langle \chi_k \mid k = 0, 1, \ldots \rangle$ with the required properties; however, any choice represents the Steenrod operator on cohomology level. The question we are asking here is whether or not it is possible to construct a particular set of $\langle \chi_k \mid k=0,1,\dots \rangle$'s so that χ_k maps the subsheaf $\mathscr{C}^*(X,\mathscr{A}) \otimes \mathscr{C}^*(X,\mathscr{A})$ into $\mathscr{C}^*(X,\mathscr{A})$ for every subsheaf $\mathscr{A}$ as above. If such a preferred system of $\langle \chi_k \mid k=0,1,\dots \rangle$'s exists, then we have that the restrictions $\left\langle \chi_k|_{[\mathscr{C}^*(X,\mathscr{A}) \otimes \mathscr{C}^*(X,\mathscr{A})]} \mid k=0,1,\dots \right\rangle$ gives us a system to construct the Steenrod operators on the subsheaf $\mathscr{A}$. In what follows, we construct such a system; thus giving an affirmative answer to the question above. For obvious reasons, we will call this preferred system *canonical.*

1.8.3 FACT

Recall that the canonical flabby resolution is point-wise homotopically trivial; i.e., for any $x \in X$, we have a splitting D_x on the stalk as

$$\mathscr{A}|_x \underset{\mu_x}{\overset{\varepsilon}{\rightleftarrows}} \mathscr{C}^0(X,\mathscr{A})|_x \underset{D_x}{\overset{d}{\rightleftarrows}} \mathscr{C}^1(X,\mathscr{A})|_x \underset{D_x}{\overset{d}{\rightleftarrows}} \cdots$$

with (1). $D_x \circ d + d \circ D_x = Id_x : \mathscr{C}^n(X,\mathscr{A})|_x \to \mathscr{C}^n(X,\mathscr{A})|_x$

(2). $D_x \circ d = Id_x - \varepsilon \circ \eta_x : \mathscr{C}^0(X,\mathscr{A})|_x \to \mathscr{C}^0(X,\mathscr{A})|_x$

(3). $D_x \circ D_x = 0_x$.

Recall that D_x is constructed as follows via the diagram

$$\begin{tikzcd}
\mathscr{L}^n(X,\mathscr{A})|_x \arrow[r, shift left, "\varepsilon"] & \mathscr{C}(\mathscr{L}^n(X,\mathscr{A}))|_x = \mathscr{C}^n(X,\mathscr{A})|_x \arrow[l, shift left, two heads, "\eta_x"] \arrow[r, shift left, two heads, "\partial"] & \mathscr{L}^{n+1}(X,\mathscr{A})|_x \arrow[l, shift left, "\pi_x"] \arrow[d, shift right, "\varepsilon"'] \\
 & & \mathscr{C}^{n+1}(X,\mathscr{A})|_x = \mathscr{C}(\mathscr{L}^{n+1}(X,\mathscr{A}))|_x \arrow[u, shift right, two heads, "\eta_x"'] \arrow[ul, "D_x"]
\end{tikzcd}$$

Take η_x to be the evaluation at x. Then $\eta_x \circ \varepsilon = \mathrm{Id}_x$. Put $\pi_x := (\mathrm{Id}_x - (\varepsilon \circ \eta_x)) \circ \partial^{-1}$; it is easy to see that π_x is well-defined (i.e. $(I_x - \varepsilon \circ \eta_x)|_{\ker \partial}$ is zero) and splits the horizontal part. Define $D_x := \pi_x \circ \eta_x$. The required properties follow immediately.

1.8.4 LEMMA

D_x is natural with respect to the subsheaf inclusion; i.e., if $\iota : \mathscr{A} \rightarrowtail \mathscr{B}$ is a subsheaf, then the following ladder commutes:

$$\begin{tikzcd}
\mathscr{B}|_x \arrow[r, shift left, tail, "\varepsilon"] & \mathscr{C}^0(X, \mathscr{B})|_x \arrow[l, shift left, two heads, "\eta_x"] \arrow[r, shift left, "d"] & \mathscr{C}^1(X, \mathscr{B})|_x \arrow[l, shift left, "D_x"] \arrow[r, shift left, "d"] & \cdots \arrow[l, shift left, "D_x"] \\
\mathscr{A}|_x \arrow[u, tail, "\iota"] \arrow[r, shift left, tail, "\varepsilon"] & \mathscr{C}^0(X, \mathscr{A})|_x \arrow[u, tail, "\iota"] \arrow[l, shift left, two heads, "\eta_x"] \arrow[r, shift left, "d"] & \mathscr{C}^1(X, \mathscr{A})|_x \arrow[u, tail, "\iota"] \arrow[l, shift left, "D_x"] \arrow[r, shift left, "d"] & \cdots \arrow[l, shift left, "D_x"]
\end{tikzcd}$$

Proof:

We can verify the above statement through a step-by-step induction applied to our previous construction of D_x. First, the evaluation η_x and the natural inclusion ε obviously commute with ι. Second, since $\partial \circ \iota = \iota \circ \partial$, we get that $\pi_x \circ \iota = \iota \circ \pi_x$. Since $D_x := \pi_x \circ \eta_x$, we get $D_x \circ \iota = \iota \circ D_x$. ■

1.8.5 NOTE

Note that another way to describe the statement in 1.8.4 is to say that the splitting $\mathscr{C}^n(X, \mathscr{B})|_x = \mathrm{Im}\, d \oplus \mathrm{Im}\, D_x$ is compatible with the filtration:

$$\mathscr{C}^n(X, \mathscr{A})|_x \subset \mathscr{C}^n(X, \mathscr{B})|_x .$$

1.8.6 DEFINITION OF Λ_x

The splitting D_x on $\mathscr{C}^*(X, \mathscr{B})$ can be lifted to the complex $\{\mathscr{C}^*(X, \mathscr{B}) \otimes \mathscr{C}^*(X, \mathscr{B}),\ d_\otimes\}$ as follows. Define

$$\Lambda_x(b_1 \otimes b_2) := (\varepsilon \circ \eta_x)(b_1) \otimes D_x(b_2) \quad \text{if } \deg b_1 = 0 \text{ and } \deg b_1 > 0\text{, and}$$

$$:= D_x(b_1) \otimes b_2 \quad \text{if } \deg b_1 > 0 \text{ and } \deg b_2\text{, any.}$$

Then routine verification shows that

(1). $\Lambda_x \circ d_\otimes + d_\otimes \circ \Lambda_x = \mathrm{Id}_x$ on $[\mathscr{C}^*(X, \mathscr{B}) \otimes \mathscr{C}^*(X, \mathscr{B})]_n$ for $n \geq 1$.

(2). $\Lambda_x \circ d_\otimes = \mathrm{Id}_x - (\varepsilon \circ \eta_x) \otimes (\varepsilon \circ \eta_x)$ on $[\mathscr{C}^*(X, \mathscr{B}) \otimes \mathscr{C}^*(X, \mathscr{B})]_n$, or, in other words, Λ_x is a pointwise splitting for the tensor product resolution. Also,

(3). $\Lambda_x \circ \Lambda_x = 0$.

1.8.7 DEFINITION OF Λ_x^1

Let Λ_x^1 be defined by

$$\Lambda_x^1 := \Lambda_x \circ \tau \circ \Lambda_x$$

where τ was defined in 1.7.2 as $\tau = 1 + \nu$, with ν being the natural "switch map".

1.8.8 LEMMA

(1). $\Lambda_x^1 \circ d_\otimes + d_\otimes \circ \Lambda_x^1 = \Lambda_x \circ \tau + \tau \circ \Lambda_x$

(2). $\Lambda_x \circ \tau \circ \Lambda_x^1 := \Lambda_x \circ \tau \circ \Lambda_x \circ \tau \circ \Lambda_x = 0$

(3). $\Lambda_x^1 \circ \tau + \tau \circ \Lambda_x^1 = 0$

on $[\mathscr{C}^*(X, \mathscr{B}) \otimes \mathscr{C}^*(X, \mathscr{B})]_n$ for $n \geq 1$.

Proof:

(1). $\Lambda_x^1 \circ d_\otimes + d_\otimes \circ \Lambda_x^1 = \Lambda_x \circ \tau \circ \Lambda_x \circ d_\otimes + d_\otimes \circ \Lambda_x \circ \tau \circ \Lambda_x$

$= \Lambda_x \circ \tau \circ (\mathrm{Id}_x + d_\otimes \circ \Lambda_x) + d_\otimes \circ \Lambda_x \circ \tau \circ \Lambda_x$

$= \Lambda_x \circ \tau + \Lambda_x \circ d_\otimes \circ \tau \circ \Lambda_x + d_\otimes \circ \Lambda_x \circ \tau \circ \Lambda_x$

$= \Lambda_x \circ \tau + \tau \circ \Lambda_x$

by 1.8.6 (1) repeatedly.

(2). A trivial consequence of $D_x \circ D_x = 0$.

(3). $\Lambda_x^1 \circ \tau + \tau \circ \Lambda_x^1 = 0$ if and only if

$(d_\otimes \circ \Lambda_x + \Lambda_x \circ d_\otimes) \circ \Lambda_x^1 \circ \tau + \tau \circ \Lambda_x^1 \circ (d_\otimes \circ \Lambda_x + \Lambda_x \circ d_\otimes) = 0$ by 1.8.6 (1).

But this latter expression is

$= d_\otimes \circ (\Lambda_x \circ \tau \circ \Lambda_x^1) + (\Lambda_x \circ \tau \circ \Lambda_x^1) \circ d_\otimes + \Lambda_x \circ \tau \circ (d_\otimes \circ \Lambda_x + \Lambda_x \circ d_\otimes) \circ \tau \circ \Lambda_x$

$= d_\otimes \circ 0 + 0 \circ d_\otimes + \Lambda_x \circ \tau \circ \tau \circ \Lambda_x = 0$. ∎

1.8.9 LEMMA

Λ_x and Λ_x^1 are natural with respect to the subsheaf inclusion; i.e., the diagram(s)

$$\begin{tikzcd}
{[\mathscr{C}^*(X, \mathscr{A}) \otimes \mathscr{C}^*(X, \mathscr{A})]|_x} \arrow[r, tail, "\iota \otimes \iota"] \arrow[d, "\Lambda_x(\Lambda_x^1)"'] & {[\mathscr{C}^*(X, \mathscr{B}) \otimes \mathscr{C}^*(X, \mathscr{B})]|_x} \arrow[d, "\Lambda_x(\Lambda_x^1)"] \\
{[\mathscr{C}^*(X, \mathscr{A}) \otimes \mathscr{C}^*(X, \mathscr{A})]|_x} \arrow[r, tail, "\iota \otimes \iota"] & {[\mathscr{C}^*(X, \mathscr{B}) \otimes \mathscr{C}^*(X, \mathscr{B})]|_x}
\end{tikzcd}$$

are commutative.

Proof:

Obviously follows from the similar statement for D_x (Lemma 1.8.4). ∎

Next we state a simple but very important lemma that is going to play a decisive role in our construction of a "canonical" system.

1.8.10 LEMMA

Let $\mathscr{A}$ and $\mathscr{B}$ be sheaves over X. Let $\{\gamma\}: \mathscr{A} \to \mathscr{B}$ be a stalk-preserving but not necessarily continuous map (i.e. a map of protosheaves).

(1). Then there exists a unique *continuous* map $\gamma: \mathscr{A} \to \mathscr{C}(X, \mathscr{B})$ such that the diagram

$$\begin{tikzcd} \mathscr{A}|_x \arrow[r, "\gamma"] \arrow[dr, "\{\gamma\}|_x"'] & \mathscr{C}(X, \mathscr{B})|_x \arrow[d, "\eta_x"] \\ & \mathscr{B}|_x \end{tikzcd}$$

commutes for any $x \in X$. *If* $\{\gamma\}$ was stalkwise in a stated category (e.g. a map of $\mathbb{Z}_2$ - algebras), then γ will be in the stated sheaf category (e.g. maps of sheaves of $\mathbb{Z}_2$ - algebras).

(2). If $\{\gamma\}$ is continuous, the γ is given by $\gamma = \varepsilon \circ \{\gamma\}$; that is, the diagram

$$\begin{tikzcd} \mathscr{A} \arrow[r, "\gamma"] \arrow[dr, "\{\gamma\}"'] & \mathscr{C}(X, \mathscr{B}) \\ & \mathscr{B} \arrow[u, "\varepsilon"'] \end{tikzcd}$$

is commutative.

(3). If $\{\gamma\}$ maps into a subsheaf $\mathscr{B}'$ of $\mathscr{B}$ then γ maps into the subsheaf $\mathscr{C}(X, \mathscr{B}')$ of $\mathscr{C}(X, \mathscr{B})$.

Proof:

The canonical flabby extension $\mathscr{C}(\{\gamma\}) : \mathscr{C}(X, \mathscr{A}) \to \mathscr{C}(X, \mathscr{B})$ is continuous. If $\varepsilon : \mathscr{A} \to \mathscr{C}(X, \mathscr{A})$ is the canonical inclusion, then the map $\gamma := \mathscr{C}(\{\gamma\}) \circ \varepsilon$ has the required properties. ∎

1.8.11 CONSTRUCTION OF THE CANONICAL SYSTEM

We construct the system $\langle \chi_k \mid k = 0, 1, \ldots \rangle$ by simultaneous induction on k and the degree of the complex $\mathscr{C}^*(X, \mathscr{B}) \otimes \mathscr{C}^*(X, \mathscr{B})$. The inductive step is essentially given by Lemma 1.8.10.

(1). If $\chi : \mathscr{B} \otimes \mathscr{B} \to \mathscr{B}$ is the multiplication on $\mathscr{B}$, then we can immediately define $\chi_0|[\mathscr{C}^0(X, \mathscr{B}) \otimes \mathscr{C}^0(X, \mathscr{B})]$ by the composition

$$\mathscr{C}^0(X, \mathscr{B}) \otimes \mathscr{C}^0(X, \mathscr{B}) \xrightarrow{\ \otimes\ } \mathscr{C}^0(X, \mathscr{B} \otimes \mathscr{B}) \xrightarrow{\ \mathscr{C}(\chi)\ } \mathscr{C}^0(X, \mathscr{B})$$

(2). Suppose that χ_0 has already been defined up to degree k . We define χ_0 on $[\mathscr{C}^*(X, \mathscr{B}) \otimes \mathscr{C}^*(X, \mathscr{B})]_{k+1}$ by the following steps:

(a). Fix $x \in X$. Take the diagram

$$\begin{array}{ccc} \mathscr{C}^k(X, \mathscr{B})|_x & \xrightarrow{\ \partial\ } & \mathscr{L}^{k+1}(X, \mathscr{B})|_x \\ & & \varepsilon \downarrow \uparrow \eta_x \\ & & \mathscr{C}(\mathscr{L}^{k+1}(X, \mathscr{B}))|_x = \mathscr{C}^{k+1}(X, \mathscr{B})|_x \end{array}$$

and define $\{\partial \circ \chi_0 \circ \Lambda_x\}|_x : [\mathscr{C}^*(X, \mathscr{B}) \otimes \mathscr{C}^*(X, \mathscr{B})]_{k+1} \to \mathscr{L}^{k+1}(X, \mathscr{B})|_x$ by the inductive supposition.

(b). Since we have a map at every point, we define a not necessarily continuous map $\{\partial \circ \chi_0 \circ \Lambda_x\} : [\mathscr{C}^*(X, \mathscr{B}) \otimes \mathscr{C}^*(X, \mathscr{B})]_{k+1} \to \mathscr{L}^{k+1}(X, \mathscr{B})$.

(c). Apply 1.8.10 to lift this map to a continuous map on serration level; this gives the definition of χ_0 on the (k+1)th degree as $\chi_0 := \partial \circ \chi_0 \circ \Lambda_x$.

(3). We have to verify that χ_0 as constructed above is indeed a chain map. Again, we proceed by induction. On $[\mathscr{C}^*(X, \mathscr{B}) \otimes \mathscr{C}^*(X, \mathscr{B})]_0$ the equality $\chi_0 \circ (\varepsilon \otimes \varepsilon) = \varepsilon \circ \chi_0$ holds trivially. Suppose now that $\chi_0 \circ d_\otimes + d \circ \chi_0 = 0$ on $[\mathscr{C}^*(X, \mathscr{B}) \otimes \mathscr{C}^*(X, \mathscr{B})]_j$ for $j \le k$. We will show the equality $\chi_0 \circ d_\otimes + d \circ \chi_0 = 0$ for degree k+1. $\chi_0 \circ d_\otimes$ is the lift of the map $\{\partial \circ \chi_0 \circ \Lambda_x \circ d_\otimes\} : [\mathscr{C}^*(X, \mathscr{B}) \otimes \mathscr{C}^*(X, \mathscr{B})]_{k+1} \to \mathscr{L}^{k+2}(X, \mathscr{B})$; while $d \circ \chi_0$ is the lift to $\{\partial \circ \chi_0\} : [\mathscr{C}^*(X, \mathscr{B}) \otimes \mathscr{C}^*(X, \mathscr{B})] \to \mathscr{L}^{k+2}(X, \mathscr{B})$ (which is $\varepsilon \circ \partial \circ \chi_0 = d \circ \chi_0$). By the uniqueness in 1.8.10, it is enough to verify that at each point x, $\{\partial \circ \chi_0 \circ \Lambda_x \circ d_\otimes\}|_x = \{\partial \circ \chi_0\}|_x$; or, $\{\chi_0 \circ \Lambda_x \circ d_\otimes + \chi_0\}|_x \in \ker \partial$, which is the same as $\{\chi_0 \circ \Lambda_x \circ d_\otimes + \chi_0\}|_x \in \operatorname{Im} \varepsilon = \operatorname{Im} d \subset \mathscr{C}^k(X, \mathscr{B})|_x$. For convenience we display the following diagram.

$$\begin{tikzcd}
{[\mathscr{C}^*(X, \mathscr{B}) \otimes \mathscr{C}^*(X, \mathscr{B})]_{k+2}} \arrow[r, "\chi_0"] \arrow[d, shift left, "\Lambda_x"] & \mathscr{C}^{k+2}(X, \mathscr{B}) \arrow[dr, shift left, "\eta_x"'] & \\
{[\mathscr{C}^*(X, \mathscr{B}) \otimes \mathscr{C}^*(X, \mathscr{B})]_{k+1}} \arrow[r, "\chi_0"] \arrow[u, shift left, "d_\otimes"] \arrow[ur, phantom, "?"] & \mathscr{C}^{k+1}(X, \mathscr{B}) \arrow[r, two heads, "\partial"'] \arrow[u, "d"] & \mathscr{L}^{k+2}(X, \mathscr{B}) \arrow[ul, tail, shift left, "\varepsilon"'] \\
{[\mathscr{C}^*(X, \mathscr{B}) \otimes \mathscr{C}^*(X, \mathscr{B})]_{k}} \arrow[r, "\chi_0"] \arrow[u, "d_\otimes"] & \mathscr{C}^{k}(X, \mathscr{B}) \arrow[r, two heads, "\partial"'] \arrow[u, "d"] & \mathscr{L}^{k+1}(X, \mathscr{B}) \arrow[ul, tail, "\varepsilon"']
\end{tikzcd}$$

where the question is the commutativity of the upper square and the induction gives the commutativity of the lower square.

Note the last containment is the consequence of the inductive supposition since

$$\begin{aligned} \chi_0 \circ \Lambda_x \circ d_\otimes + \chi_0 &= \chi_0 + \chi_0 + \chi_0 \circ d_\otimes \circ \Lambda_x \\ &= \chi_0 \circ d_\otimes \circ \Lambda_x \\ &= d \circ \chi_0 \circ \Lambda_x \in \operatorname{Im} d . \end{aligned}$$

(4). Now we have χ_0 defined as a chain map extending χ. Next we shall use double induction to define χ_k. The inductive step is again supplied by 1.8.10.

(a). First, define $\chi_k := 0$ on $[\mathscr{C}^*(X, \mathscr{B}) \otimes \mathscr{C}^*(X, \mathscr{B})]_j$ for $0 \le j \le k-1$. Suppose now that the system $\langle \chi_0, \chi_1, \cdots, \chi_{k-1} \rangle$ has already been defined on $\mathscr{C}^*(X, \mathscr{A}) \otimes \mathscr{C}^*(X, \mathscr{A})$ and χ_k has been given on $[\mathscr{C}^*(X, \mathscr{B}) \otimes \mathscr{C}^*(X, \mathscr{B})]_j$ for $j \le n$. Under this supposition,

(b). define χ_k on $[\mathscr{C}^*(X, \mathscr{B}) \otimes \mathscr{C}^*(X, \mathscr{B})]_{n+1}$ to be the (not necessarily continuous) map $\{\partial \circ \chi_k \circ \Lambda_x + \partial \circ \chi_{k-1} \circ \Lambda_x^1\} : [\mathscr{C}^*(X, \mathscr{B}) \otimes \mathscr{C}^*(X, \mathscr{B})]_{n+1} \to \mathscr{L}^{n-k+1}(X, \mathscr{B})$. For convenience, the reader may refer to the diagram below.

$$\begin{tikzcd}
{[\mathscr{C}^*(X, \mathscr{B}) \otimes \mathscr{C}^*(X, \mathscr{B})]_{n+1}} \arrow[r, "\chi_k"] \arrow[d, dashed, shift right, "\Lambda_x"'] \arrow[dd, dashed, bend right=60, "\Lambda_x^1"'] \arrow[drr, dashed, "\psi"] & \mathscr{C}^{n-k+1}(X, \mathscr{B}) \arrow[dr, shift left, "\varepsilon"] & \\
{[\mathscr{C}^*(X, \mathscr{B}) \otimes \mathscr{C}^*(X, \mathscr{B})]_{n}} \arrow[r, "\chi_k"'] \arrow[u, shift right, "d"'] & \mathscr{C}^{n-k}(X, \mathscr{B}) \arrow[r, two heads, "\partial"'] \arrow[u, "d"] & \mathscr{L}^{n-k+1}(X, \mathscr{B}) \arrow[ul, dashed, shift left, "\eta_x"] \\
{[\mathscr{C}^*(X, \mathscr{B}) \otimes \mathscr{C}^*(X, \mathscr{B})]_{n-1}} \arrow[u, "d"] \arrow[ur, "\chi_{k-1}"'] & &
\end{tikzcd}$$

where the symbol ψ stands for the map $\psi := \{\partial \circ \chi_{k-1} \circ \Lambda_x^1\}$.

(5). By the previous inductive construction, we obtained a system $\langle \chi_k \mid k = 0, 1, \ldots \rangle$ with the property that χ_k vanishes on the first (k-1) degrees and χ_0 is a chain map extending χ. Now we verify the property $\chi_k \circ d_\otimes + d \circ \chi_k = \chi_{k-1} \circ \tau$, and thus check that $\langle \chi_k \mid k = 0, 1, \ldots \rangle$ is actually a system of maps that induces the Steenrod operations on cohomology level.

By definition the equality above holds for χ_0 ($\chi_{k-1} = 0$ by convention) and for any k up to $[\mathscr{C}^*(X, \mathscr{B}) \otimes \mathscr{C}^*(X, \mathscr{B})]_{k-1}$. We again use simultaneous induction. Suppose the equality holds for $\langle \chi_0, \chi_1, \cdots, \chi_{k-1} \rangle$ and for χ_k in degrees $j = 0, 1, \ldots, n-1$. Under this supposition we prove the formula for χ_k on $[\mathscr{C}^*(X, \mathscr{B}) \otimes \mathscr{C}^*(X, \mathscr{B})]_n$. Our argument is completely analogous to the one in (3), namely:

(a). $\chi_k \circ d_\otimes$ is the lift of the map $\{\partial \circ \chi_k \circ \Lambda_x \circ d_\otimes + \partial \circ \chi_{k-1} \circ \Lambda_x^1 \circ d_\otimes\}$: $[\mathscr{C}^*(X, \mathscr{B}) \otimes \mathscr{C}^*(X, \mathscr{B})]_n \to \mathscr{L}^{\,n-k+1}(X, \mathscr{B})$, and $d \circ \chi_k$ lifts the map $\{\partial \circ \chi_k\} : [\mathscr{C}^*(X, \mathscr{B}) \otimes \mathscr{C}^*(X, \mathscr{B})]_n \to \mathscr{L}^{\,n-k+1}(X, \mathscr{B})$,
while $\chi_{k-1} \circ \tau$ is given as a lift of $\{\partial \circ \chi_{k-1} \circ \Lambda_x \circ \tau + \partial \circ \chi_{k-2} \circ \Lambda_x^1 \circ \tau\}$: $[\mathscr{C}^*(X, \mathscr{B}) \otimes \mathscr{C}^*(X, \mathscr{B})]_n \to \mathscr{L}^{\,n-k+1}(X, \mathscr{B})$. By the unicity of the lifting, it is sufficient to verify the equality

(b).
$$\{\partial \circ \chi_k \circ \Lambda_x \circ d_\otimes + \partial \circ \chi_{k-1} \circ \Lambda_x^1 \circ d_\otimes + \partial \circ \chi_k\}$$
$$= \{\partial \circ \chi_{k-1} \circ \Lambda_x \circ \tau + \partial \circ \chi_{k-2} \circ \Lambda_x^1 \circ \tau\}$$

which amounts to the statement that
$$\{\chi_k \circ \Lambda_x \circ d_\otimes + \chi_{k-1} \circ \Lambda_x^1 \circ d_\otimes + \chi_k\}$$
$$\equiv \{\chi_{k-1} \circ \Lambda_x \circ \tau + \chi_{k-2} \circ \Lambda_x^1 \circ \tau\} \bmod (\mathrm{Im}\, d)$$

by exactness. Here, "$\equiv$ mod (Im d)" stands for the statement that the sides are equivalent in $\mathscr{C}^{\,n-k}(X, \mathscr{B})|_x$ modulo the subgroup $(\mathrm{Im}\, d)|_x$ for every $x \in X$.

We prove the stated equivalence by exchanging the terms on the left as follows. First, the fact that Λ_x is the homotopy splitting map implies that

$\{\chi_k \circ \Lambda_X \circ d_\otimes\} = \{\chi_k + \chi_k \circ d_\otimes \circ \Lambda_X\}$. Second, by Lemma 1.8.8 we can write

$\{\chi_{k-1} \circ \Lambda_X^1 \circ d_\otimes\} = \{\chi_{k-1} \circ d_\otimes \circ \Lambda_X^1 + \chi_{k-1} \circ \Lambda_X \circ \tau + \chi_{k-1} \circ \tau \circ \Lambda_X\}$.

Furthermore, making use of our inductive supposition, we get

$\{\chi_k \circ d_\otimes \circ \Lambda_X\} = \{d \circ \chi_k \circ \Lambda_X + \chi_{k-1} \circ \tau \circ \Lambda_X\} \equiv \{\chi_{k-1} \circ \tau \circ \Lambda_X\} \bmod (\mathrm{Im}\, d)$;

$\{\chi_{k-1} \circ d_\otimes \circ \Lambda_X^1\} = \{d \circ \chi_{k-1} \circ \Lambda_X^1 + \chi_{k-2} \circ \tau \circ \Lambda_X^1\} \equiv \{\chi_{k-2} \circ \tau \circ \Lambda_X^1\} \bmod (\mathrm{Im}\, d)$.

Now substitution yields the following string of equalities.

$$\begin{aligned}
&\{\chi_k \circ \Lambda_X \circ d_\otimes + \chi_{k-1} \circ \Lambda_X^1 \circ d_\otimes + \chi_k\} \\
&\equiv \{\chi_k + \chi_{k-1} \circ \tau \circ \Lambda_X + \chi_{k-2} \circ \tau \circ \Lambda_X^1 + \chi_{k-1} \circ \Lambda_X \circ \tau + \chi_{k-1} \circ \tau \circ \Lambda_X + \chi_k\} \\
&= \{\chi_{k-2} \circ \Lambda_X^1 \circ \tau + \chi_{k-1} \circ \Lambda_X \circ \tau\} \bmod (\mathrm{Im}\, d)
\end{aligned}$$

where the last equality holds by 1.8.8 (3).

This finishes the proof of the fact that our canonical system satisfies the properties required in the previous chapter.

1.8.12

The canonical system $\langle \chi_k \mid k = 0, 1, \ldots \rangle$ has many desirable features. In the next lemma, we prove that the canonical system directly respects subsheaves and thus gives the affirmative answer to the question posed at the beginning of the section. Then, following the lemma, we will find a convenient expression for $Sq^0 : H^n(X, \mathscr{B}) \to H^n(X, \mathscr{B})$. We also include a remark about the nonexistence of Sq^k for $k < 0$; this later result relies heavily on the canonical construction above.

1.8.13 LEMMA

Let $\mathscr{A} \rightarrowtail \mathscr{B}$ be a subsheaf of Z_2 - algebras. Let $\langle \chi_k \mid k = 0, 1, \ldots \rangle$ be the canonical system constructed for the sheaf $\mathscr{B}$ as in 1.8.11. Then χ_k maps the subsheaf $\mathscr{C}^*(X, \mathscr{A}) \otimes \mathscr{C}^*(X, \mathscr{A})$ into the subsheaf $\mathscr{C}^*(X, \mathscr{A})$ of $\mathscr{C}^*(X, \mathscr{B})$.

<u>Proof</u>:

$\chi_0|[\mathscr{C}^*(X,\mathscr{A})\otimes\mathscr{C}^*(X,\mathscr{A})]_0$ and

$\chi_k|[\mathscr{C}^*(X,\mathscr{A})\otimes\mathscr{C}^*(X,\mathscr{A})]_j$ $j = 0, 1, \ldots, k\text{-}1$

obviously map into $\mathscr{C}^*(X, \mathscr{A})$. The proof now goes by induction and is very much the same as the argument in 1.8.11 (5). The key observation is that the subsheaf $\mathscr{C}^*(X, \mathscr{A}) \otimes \mathscr{C}^*(X, \mathscr{A})$ is invariant under the operators Λ_x and Λ_x^1 (see Lemma 1.8.9). Again, the proof is divided into two steps.

(a). Under the inductive supposition "χ_0 maps $[\mathscr{C}^*(X, \mathscr{A}) \otimes \mathscr{C}^*(X, \mathscr{A})]_n$ into $\mathscr{C}^n(X, \mathscr{A})$", we conclude that $\{\partial \circ \chi_0 \circ \Lambda_x\}$ maps $[\mathscr{C}^*(X, \mathscr{A}) \otimes \mathscr{C}^*(X, \mathscr{A})]_{n+1}$ into $\mathscr{L}^{n+1}(X, \mathscr{A}) \rightarrowtail \mathscr{L}^{n+1}(X, \mathscr{B})$. By the fact that the lifting respects subsheaves (see 1.8.10 (3)), we make the inductive step and thus obtain the statement for χ_0.

(b). We suppose the validity of the statement for $\langle \chi_0, \chi_1, \cdots, \chi_{k-1} \rangle$ and for χ_k in degrees less than n. Our supposition and the above mentioned invariance with respect to the operators Λ_x and Λ_x^1 implies that $\{\partial \circ \chi_k \circ \Lambda_x + \partial \circ \chi_{k-1} \circ \Lambda_x^1\}$ maps $[\mathscr{C}^*(X, \mathscr{A}) \otimes \mathscr{C}^*(X, \mathscr{A})]_{n+1}$ into the subsheaf $\mathscr{L}^{n+1-k}(X, \mathscr{A}) \rightarrowtail \mathscr{L}^{n+1-k}(X, \mathscr{B})$. Then once again we complete the inductive step by 1.8.10 (3).

■

1.8.14 LEMMA

Suppose $2k > n$. Then $\chi_k|_{[\mathscr{C}^*(X,\mathscr{B})\otimes\mathscr{C}^*(X,\mathscr{B})]_n} = 0$ where $\langle \chi_k \mid k = 0, 1, \ldots \rangle$ is the canonical system.

Proof:

Induction as in 1.8.13 or 1.8.11 gives the result. ∎

1.8.15 LEMMA

Let $\langle \psi_k : \mathscr{C}^k(X, \mathscr{B}) \to \mathscr{C}^k(X, \mathscr{B}) \mid k = 0, 1, \ldots \rangle$ be defined by $\psi_k := \chi_k \circ \theta$. Then $\langle \psi_k \mid k = 0, 1, \ldots \rangle$ gives a map of differential graded sheaves; that is, $d \circ \psi_k = \psi_{k+1} \circ d$.

Proof:

Let $c \in \mathscr{C}^k(X, \mathscr{B})|_x$.

$$\begin{aligned} \text{Then } (\psi_{k+1} \circ d)(c) := \chi_{k+1}(dc \otimes dc) &= (\chi_{k+1} \circ d_\otimes)(c \otimes dc) \\ &= d_\otimes \circ \chi_{k+1}(c \otimes dc) + \chi_k(c \otimes dc + dc \otimes c) \\ &= 0 + (\chi_k \circ d_\otimes)(c \otimes c) \\ &= (d \circ \chi_k)(c \otimes c) := d \circ \psi_k(c) \end{aligned}$$

where we used $\chi_{k+1}(c \otimes dc) = 0$, which is justified by $2(k+1) > k+(k+1) = 2k+1$ and the previous lemma.

1.8.16

ψ_k evidently extends the map given by "squaring" stalkwise on $\mathscr{B}$. Since any two chain map extensions are chain homotopically equivalent, Lemma 1.8.15 gives us a convenient method to compute Sq^0 via any chain map extension of the "squaring" operation viewed as a map from $\mathscr{B}$ to $\mathscr{B}$.

1.8.17 LEMMA

$$Sq^j = 0 \text{ for } j < 0.$$

Proof:

Evident consequence of 1.8.14. ■

CHAPTER II

2.1. STEENROD SQUARE AND TRANSFER

In this section, we formulate the sheaf - theoretical version of Bott's commutator formula. The proof will be carried out in section 2.2.

2.1.1

In sections 1.7 thru 1.8 we have established several key facts about Steenrod squares. In 1.5 we examined the transfer map and its basic properties. Here we place some of the results in the context in which we later wish to use them and also pose a question which will motivate our further discussion.

Let (X, a) be a space with involution, Y and $p : X \to Y$, L and ℓ defined as in 1.5.1. Let $\mathscr{B}$ be a sheaf on Y. We take the relative sheaf $(\overleftarrow{p}\mathscr{B})_{\tilde{L}}$ representing the relative cohomology $H^*(X, L, \overleftarrow{p}\mathscr{B})$ (see 1.5.20). If

$$p_L^+ : (\overleftrightarrow{p}\mathscr{B})_{\tilde{\ell}} \cong \vec{p}((\overleftarrow{p}\mathscr{B})_{\tilde{L}}) \to (\overleftarrow{p}\mathscr{B})_{\tilde{L}}$$

is the natural cohomomorphism between the push forward and the original sheaves, then p_L^+ induces an cohomology isomorphism (see 1.2.11, Vietoris - Begle theorem). Further, if we equip $(\overleftrightarrow{p}\mathscr{B})_{\tilde{\ell}} \cong \vec{p}((\overleftarrow{p}\mathscr{B})_{\tilde{L}})$ with the pushed forward $\mathcal{Z}_2$ - algebra structure, then the following diagram commutes by 1.7.13.

$$\begin{tikzcd}
H^i(Y, (\overleftrightarrow{p}\mathscr{B})_{\tilde{\ell}}) = H^i(Y, \ell, \overleftrightarrow{p}\mathscr{B}) \arrow[r, "St_k"] \arrow[d, "p_L^+"', "\cong"] & H^{2i-k}(Y, \ell, \overleftrightarrow{p}\mathscr{B}) \equiv H^{2i-k}(Y, (\overleftrightarrow{p}\mathscr{B})_{\tilde{\ell}}) \arrow[d, "p_L^+"', "\cong"] \\
H^i(X, (\overleftarrow{p}\mathscr{B})_{\tilde{L}}) = H^i(X, L, \overleftarrow{p}\mathscr{B}) \arrow[r, "St_k"'] & H^{2i-k}(X, L, \overleftarrow{p}\mathscr{B}) \equiv H^{2i-k}(X, (\overleftarrow{p}\mathscr{B})_{\tilde{L}})
\end{tikzcd}$$

As in 1.5.9, the involution a naturally defines the map $\sigma_L : (\overleftrightarrow{p}\mathscr{B})_{\tilde{\ell}} \twoheadrightarrow (\mathscr{B})_{\tilde{\ell}}$. The composition of the maps on cohomology level ($(p_L^+)^{-1} : H^i(X, L, \overleftarrow{p}\mathscr{B}) \to H^i(Y, \ell, \mathscr{B})$ and $\sigma_L : H^i(Y, \ell, \overleftrightarrow{p}\mathscr{B}) \to H^i(Y, \ell, \mathscr{B})$) yields the map $\Delta_L : H^i(X, L, \overleftarrow{p}\mathscr{B}) \to H^i(Y, \ell, \mathscr{B})$, which is called *the transfer* (see 1.5.15).

The map Δ_L is part of a long exact sequence

$$\begin{tikzcd}
& H^*(X, (\overleftarrow{p}\mathscr{B})_{\tilde{L}}) = H^*(X, L, \overleftarrow{p}\mathscr{B}) \arrow[dr, "\Delta_L"] & \\
H^*(Y, \ell, \mathscr{B}) = H^*(Y, (\mathscr{B})_{\tilde{\ell}}) \arrow[ur, "p_L^*"] & & H^*(Y, (\mathscr{B})_{\tilde{\ell}}) = H^*(Y, \ell, \mathscr{B}) \arrow[ll, "\mu_L"]
\end{tikzcd}$$

with connecting map μ_L (Smith sequence).

The long exact sequence above is generated by the short exact sequence of sheaves

$$(\mathscr{B})_{\tilde{\ell}} \overset{\nabla_L}{\rightarrowtail} (\overleftrightarrow{p}\mathscr{B})_{\tilde{\ell}} \overset{\sigma_L}{\twoheadrightarrow} (\mathscr{B})_{\tilde{\ell}}$$

whose stalkwise description is particularly simple. Following [Bott] as we outlined in 1.1.3, we may ask if there is some expression for the commutator

$$St_k \circ \Delta_L + \Delta_L \circ St_k : H^n(X, L, \overleftarrow{p}\mathscr{B}) \to H^{2n-k}(Y, \ell, \mathscr{B})$$

in the following diagram.

$$\begin{tikzcd}
H^*(X, L, \overleftarrow{p}\mathscr{B}) = H^*(X, (\overleftarrow{p}\mathscr{B})_{\tilde{L}}) \arrow[r, "St^k"] \arrow[d, "\Delta_L"'] & H^*(X, (\overleftarrow{p}\mathscr{B})_{\tilde{L}}) = H^*(X, L, \overleftarrow{p}\mathscr{B}) \arrow[d, "\Delta_L"] \\
H^*(Y, \ell, \mathscr{B}) = H^*(Y, (\mathscr{B})_{\tilde{\ell}}) \arrow[r, "St^k"'] & H^*(Y, (\mathscr{B})_{\tilde{\ell}}) = H^*(Y, \ell, \mathscr{B})
\end{tikzcd}$$

In fact, the answer turns out to be entirely analogous to the simplicial case in **[Bott]**, namely:

2.1.2. THEOREM [BOTT'S FORMULA]

$$St_k \circ \Delta_L + \Delta_L \circ St_k = \mu_L \circ St_{k+1} \circ \Delta_L$$

or, equivalently,

$$Sq^k \circ \Delta_L + \Delta_L \circ Sq^k = \mu_L \circ Sq^{k-1} \circ \Delta_L$$

in terms of the Steenrod squares.

<u>Proof</u>: See part 2.2. ■

2.1.3. REMARK

Our intuition suggests that the commutator above is actually zero. This is motivated by the fact that we took the relative cohomology and thus we "excised" the fixed point set in some sense. In section 2.3, we make this observation more precise, and prove

$$\mathrm{St}_k \circ \Delta_L + \Delta_L \circ \mathrm{St}_k = 0$$

by induction. However, the inductive step in the proof will be supplied by the formula given in Theorem 2.1.2.

2.1.4 REMARK

Since the diagram

$$\begin{tikzcd}
 & H^i(Y,\ell,\overleftrightarrow{p}\mathcal{S}) \arrow[r, "\mathrm{St}_k"] \arrow[dd, "\sigma_L"] & H^{2i-k}(Y,\ell,\overleftrightarrow{p}\mathcal{S}) \arrow[dd, "\sigma_L"'] & \\
H^i(X,L,\overleftarrow{p}\mathcal{S}) \arrow[ur, "(p_L^+)^{-1}"] \arrow[dr, "\Delta_L"'] \arrow[rrr, "\mathrm{St}_k"] & & & H^{2i-k}(X,L,\overleftarrow{p}\mathcal{S}) \arrow[ul, "(p_L^+)^{-1}"'] \\
 & H^i(Y,\ell,\mathcal{S}) \arrow[r, "\mathrm{St}_k"] & H^{2i-k}(Y,\ell,\mathcal{S}) \arrow[ur, "\Delta_L"'] &
\end{tikzcd}$$

has commutative top square and side triangles, it is sufficient to compute the commutator

$$\sigma_L \circ \mathrm{St}_k + \mathrm{St}_k \circ \sigma_L : H^i(Y,\ell,\overleftrightarrow{p}\mathcal{S}) \to H^{2i-k}(Y,\ell,\mathcal{S}) .$$

This is the exact strategy we are going to follow in section 2.2.

2.2 THE PROOF OF THE COMMUTATOR FORMULA

In this section, we carry out the proof of Bott's formula.

2.2.1

We compute the commutator as we outlined in 2.1.4. Let $\alpha : (\overleftrightarrow{p}\mathscr{B})_{\tilde{\ell}} \to (\overleftrightarrow{p}\mathscr{B})_{\tilde{\ell}}$ be the stalkwise "switch" as in 1.5.9. We wish to extend α to the canonical flabby resolution. This can be done in a highly plausible way, namely:

2.2.2 LEMMA

There exists an extension $\overline{\alpha} : \mathscr{C}^*(Y, (\overleftrightarrow{p}\mathscr{B})_{\tilde{\ell}}) \to \mathscr{C}^*(Y, (\overleftrightarrow{p}\mathscr{B})_{\tilde{\ell}})$ such that the following ladder commutes:

$$\begin{tikzcd}
(\overleftrightarrow{p}\mathscr{B})_{\tilde{\ell}} \arrow[r, shift left, "\varepsilon"] \arrow[d, "\alpha"'] & \mathscr{C}^0(Y, (\overleftrightarrow{p}\mathscr{B})_{\tilde{\ell}}) \arrow[l, shift left, twoheadrightarrow, "\eta_x"] \arrow[r, shift left, "d"] \arrow[d, "\overline{\alpha}"'] & \mathscr{C}^1(Y, (\overleftrightarrow{p}\mathscr{B})_{\tilde{\ell}}) \arrow[l, shift left, "D_x"] \arrow[d, "\overline{\alpha}"'] & \cdots \\
(\overleftrightarrow{p}\mathscr{B})_{\tilde{\ell}} \arrow[r, shift left, "\varepsilon"] & \mathscr{C}^0(Y, (\overleftrightarrow{p}\mathscr{B})_{\tilde{\ell}}) \arrow[l, shift left, twoheadrightarrow, "\eta_x"] \arrow[r, shift left, "d"] & \mathscr{C}^1(Y, (\overleftrightarrow{p}\mathscr{B})_{\tilde{\ell}}) \arrow[l, shift left, "D_x"] & \cdots ,
\end{tikzcd}$$

and such that $\overline{\alpha} \circ \overline{\alpha} = \mathrm{Id}$.

<u>Proof</u>:

By induction, it is obvious that the canonical flabby extension $\overline{\alpha} = \mathscr{C}(\alpha)$ satisfies the requirements. ∎

2.2.3

Extend $\overline{\alpha} : \mathscr{C}^*(Y, (\overleftrightarrow{p}\mathscr{B})_{\tilde{\jmath}}) \to \mathscr{C}^*(Y, (\overleftrightarrow{p}\mathscr{B})_{\tilde{\jmath}})$ to the tensor product as

$$\overline{\alpha} \otimes \overline{\alpha} : \mathscr{C}^*(Y, (\overleftrightarrow{p}\mathscr{B})_{\tilde{\jmath}}) \otimes \mathscr{C}^*(Y, (\overleftrightarrow{p}\mathscr{B})_{\tilde{\jmath}}) \to \mathscr{C}^*(Y, (\overleftrightarrow{p}\mathscr{B})_{\tilde{\jmath}}) \otimes \mathscr{C}^*(Y, (\overleftrightarrow{p}\mathscr{B})_{\tilde{\jmath}}) .$$

Obviously, $\tau \circ (\overline{\alpha} \otimes \overline{\alpha}) = (\overline{\alpha} \otimes \overline{\alpha}) \circ \tau$, since $\overline{\alpha}$ is "internal" while τ is "external" on the stalks. We also have $\Lambda_x \circ (\overline{\alpha} \otimes \overline{\alpha}) = (\overline{\alpha} \otimes \overline{\alpha}) \circ \Lambda_x$ as a consequence of 2.2.2; i.e., $D_x \circ \overline{\alpha} = D_x \circ \overline{\alpha}$. This immediately yields $\Lambda^1_x \circ (\overline{\alpha} \otimes \overline{\alpha}) = (\overline{\alpha} \otimes \overline{\alpha}) \circ \Lambda^1_x$.

Next, we shall prove the formula $\overline{\alpha} \circ \chi_k = \chi_k \circ (\overline{\alpha} \otimes \overline{\alpha})$, where $\langle \chi_k \mid k = 0, 1, \ldots \rangle$ is the canonical system defined in section 1.7. To carry out the induction, we need the following trivial lemma:

2.2.4 LEMMA

In Lemma 1.7.10, let $\beta : \mathscr{B} \to \mathscr{B}$ and $\alpha : \mathscr{A} \to \mathscr{A}$ be sheaf homomorphisms such that $\{\gamma\} \circ \alpha = \beta \circ \{\gamma\}$. Let $\overline{\beta} = \mathscr{C}(\beta)$ be the canonical extension of β to $\mathscr{C}(\mathscr{B})$. Then $\gamma \circ \alpha = \overline{\beta} \circ \gamma$, i.e.,

$$\begin{tikzcd}
\mathscr{A} \arrow[loop left, "\alpha"] \arrow[r, "\gamma"] \arrow[dr, "\{\gamma\}"'] & \mathscr{C}(\mathscr{B}) \arrow[loop right, "\overline{\beta}"] \arrow[d, "\eta_x"] \\
 & \mathscr{B} \arrow[loop right, "\beta"]
\end{tikzcd}$$

commutes.

<u>Proof</u>: $\gamma \circ \alpha$ and $\overline{\beta} \circ \gamma$ both lift $\beta \circ \{\gamma\}$, so by unicity, they must agree. ∎

2.2.5 THEOREM

Let $\langle \chi_k \mid k = 0, 1, \ldots \rangle$ be the canonical system constructed for $(\overleftrightarrow{p}\mathscr{B})_{\tilde{\imath}}$. Let $\overline{\alpha}$ and $(\overline{\alpha} \otimes \overline{\alpha})$ be as above. Then $\overline{\alpha} \circ \chi_k = \chi_k \circ (\overline{\alpha} \otimes \overline{\alpha})$, i.e., the following diagram commutes:

$$\begin{tikzcd}
\mathscr{C}^*(Y, (\overleftrightarrow{p}\mathscr{B})_{\tilde{\imath}}) \otimes \mathscr{C}^*(Y, (\overleftrightarrow{p}\mathscr{B})_{\tilde{\imath}}) \arrow[r, "\chi_k"] \arrow[d, "(\overline{\alpha} \otimes \overline{\alpha})"] & \mathscr{C}^*(Y, (\overleftrightarrow{p}\mathscr{B})_{\tilde{\imath}}) \arrow[d, "\overline{\alpha}"] \\
\mathscr{C}^*(Y, (\overleftrightarrow{p}\mathscr{B})_{\tilde{\imath}}) \otimes \mathscr{C}^*(Y, (\overleftrightarrow{p}\mathscr{B})_{\tilde{\imath}}) \arrow[r, "\chi_k"] & \mathscr{C}^*(Y, (\overleftrightarrow{p}\mathscr{B})_{\tilde{\imath}})
\end{tikzcd} .$$

<u>Proof</u>:

(a). First, we have commutativity on the augmentation level :

$$\begin{tikzcd}
(a_1, a_2) \otimes (b_1, b_2) \arrow[d, "\chi"] \arrow[dr, "(\alpha \otimes \alpha)"] & \\
(a_1 \cdot b_1,\ a_2 \cdot b_2) \arrow[d, "\alpha"] & (a_2, a_1) \otimes (b_2, b_1) \arrow[dl, "\chi"] \\
(a_2 \cdot b_2,\ a_1 \cdot b_1) &
\end{tikzcd}$$

And, in fact, this trivial diagram is the key to the whole argument.

(b). Since χ lifts to $\mathscr{C}^*(Y, (\overleftrightarrow{p}\mathscr{B})_{\bar{\imath}}) \otimes \mathscr{C}^*(Y, (\overleftrightarrow{p}\mathscr{B})_{\bar{\imath}})$ by pointwise multiplication of serrations, we get the commutativity for χ_0 in degree zero.

(c). We proceed by induction on the degree:

$$\begin{aligned} \partial \circ \chi_0 \circ \Lambda_x \circ (\bar{\alpha} \otimes \bar{\alpha}) &= \partial \circ \chi_0 \circ (\bar{\alpha} \otimes \bar{\alpha}) \circ \Lambda_x && \text{by } 2.2.3 \\ &= \partial \circ \bar{\alpha} \circ \chi_0 \circ \Lambda_x && \text{by induction} \\ &= \bar{\alpha} \circ \partial \circ \chi_0 \circ \Lambda_x && \text{by } 2.2.2. \end{aligned}$$

Then by 2.2.4, we get the inductive step.

(d). For χ_k, $k > 0$, we proceed by simultaneous induction. Since the commutation is true for χ_0 and the first k degrees for χ_k, we merely repeat usual steps, using 2.2.3 and 2.2.2 again. 2.2.4 gives the inductive step as usual. ■

2.2.6 LEMMA

Let $\bar{\sigma}_L := 1 + \bar{\alpha} : \mathscr{C}^*(Y, (\overleftrightarrow{p}\mathscr{B})_{\bar{\imath}}) \to \mathscr{C}^*(Y, (\overleftrightarrow{p}\mathscr{B})_{\bar{\imath}})$.

Let $\langle \chi_k \mid k = 0, 1, \ldots \rangle$ as above.

Then $(\bar{\sigma}_L \circ \chi_i)(c \otimes \bar{\sigma}_L c) = \chi_i(\bar{\sigma}_L c \otimes \bar{\sigma}_L c) = (\bar{\sigma}_L \circ \chi_i)(\bar{\sigma}_L c \otimes c)$,

where $c \in \mathscr{C}^*_Y(Y, (\overleftrightarrow{p}\mathscr{B})_{\bar{\imath}})$, (any).

<u>Proof:</u>

$$(\bar{\sigma}_L \circ \chi_i)(c \otimes \bar{\sigma}_L c) = \chi_i(c \otimes c) + \chi_i(c \otimes \bar{\alpha}c) + (\bar{\alpha} \circ \chi_i)(c \otimes c) + (\bar{\alpha} \circ \chi_i)(c \otimes \bar{\alpha}c)$$

$$\chi_i(\bar{\sigma}_L c \otimes \bar{\sigma}_L c) = \chi_i(c \otimes c) + \chi_i(c \otimes \bar{\alpha}c) + \chi_i(\bar{\alpha}c \otimes c) + \chi_i(\bar{\alpha}c \otimes \bar{\alpha}c)$$

and $(\bar{\alpha} \circ \chi_i)(c \otimes c) = \chi_i(\bar{\alpha}c \otimes \bar{\alpha}c)$, $(\bar{\alpha} \circ \chi_i)(c \otimes \bar{\alpha}c) = \chi_i(\bar{\alpha}c \otimes c)$;

by 2.2.5 and $\bar{\alpha} \circ \bar{\alpha} = \mathrm{Id}$; this proves the first equality.

The second equality follows exactly the same way. ∎

2.2.7

The short exact sequence

$$(\mathscr{B})_{\tilde{\ell}} \xrightarrow{\nabla_L} (\overleftrightarrow{p}\mathscr{B})_{\tilde{\ell}} \xrightarrow{\bar{\sigma}_L} \operatorname{Im}\bar{\sigma}_L \equiv (\mathscr{B})_{\tilde{\ell}}$$

extends to a short exact sequence

$$\mathscr{C}^*(Y, (\mathscr{B})_{\tilde{\ell}}) \xrightarrow{\bar{\nabla}_L} \mathscr{C}^*(Y, (\overleftrightarrow{p}\mathscr{B})_{\tilde{\ell}}) \xrightarrow{\bar{\sigma}_L} \mathscr{C}^*(Y, (\mathscr{B})_{\tilde{\ell}}),$$

which gives rise to the Smith sequence. By $\bar{\nabla}_L$, we will view $\mathscr{C}^*(Y, (\mathscr{B})_{\tilde{\ell}})$ as a subsheaf of $\mathscr{C}^*(Y, (\overleftrightarrow{p}\mathscr{B})_{\tilde{\ell}})$; in fact, it is both the kernel and the cokernel of $\bar{\sigma}_L$. By the canonical construction of the $\langle \chi_k \mid k = 0, 1, \dots \rangle$ system, we can perform every computation in the sheaf $\mathscr{C}^*(Y, (\overleftrightarrow{p}\mathscr{B})_{\tilde{\ell}})$.

Now we proceed as follows. Let $c \in \mathscr{C}^n_Y(Y, (\overleftrightarrow{p}\mathscr{B})_{\tilde{\ell}})$ be a cocycle representing some cohomology class $[c]$ in $H^n(Y, (\overleftrightarrow{p}\mathscr{B})_{\tilde{\ell}}) \equiv H^n(X, L)$. We give an explicit representative for the elements $St_i \circ \Delta_L[c]$, $\Delta_L \circ St_i[c]$, $\mu_L \circ St_{i+1} \circ \Delta_L[c]$ in 2.1.2.

This goes as follows:

$St_i \circ \Delta_L[c]$ is represented by $\chi_i \circ (\bar{\sigma}_L c \otimes \bar{\sigma}_L c)$

$\Delta_L \circ St_i[c]$ is represented by $(\bar{\sigma}_L \circ \chi_i)(c \otimes c)$

and finally,

2.2.8 LEMMA

$\mu_L \circ St_{i+1} \circ \Delta_L[c]$ is represented by $(d \circ \chi_{i+1})(c \otimes \bar{\alpha} c)$.

Proof:

$St_{i+1} \circ \Delta_L[c]$ is represented by $\chi_{i+1}(\bar{\sigma}_L c \otimes \bar{\sigma}_L c)$ since χ_i was the canonical system (i.e. χ_j restricted to $\mathscr{C}^*(Y, (\mathscr{B})_{\tilde{l}}) \otimes \mathscr{C}^*(Y, (\mathscr{B})_{\tilde{l}})$ gives the Steenrod operation on $H^*(Y, l, \mathscr{B})$).

Now we compute the connecting μ_L.

By 2.2.6, $(\bar{\sigma}_L \circ \chi_{i+1})(c \otimes \bar{\sigma}_L c) = \chi_{i+1}(\bar{\sigma}_L c \otimes \bar{\sigma} c_L)$; i.e., the section $\chi_{i+1}(c \otimes \bar{\sigma}_L c)$ maps onto $\chi_i(\bar{\sigma}_L c \otimes \bar{\sigma}_L c)$ by $\bar{\sigma}_L$, which implies that $(d \circ \chi_{i+1})(c \otimes \bar{\sigma}_L c)$ is an element in $\mathscr{C}_Y^{2n-i}(Y, (\mathscr{B})_{\tilde{l}})$ and it is a representative cycle for $\mu_L \circ St_{i+1} \circ \Delta_L[c]$.

But $(d \circ \chi_{i+1})(c \otimes \bar{\sigma}_L c) = (d \circ \chi_{i+1})(c \otimes tc) + (d \circ \chi_{i+1})(c \otimes c)$ and since c was a cocycle, we have $(d \circ \chi_{i+1})(c \otimes c) = 0$. ■

2.2.9 LEMMA

Let c be as in 2.2.7.

Then $\chi_i \circ (\bar{\sigma}_L c \otimes \bar{\sigma}_L c) + (\bar{\sigma}_L \circ \chi_i)(c \otimes c) = (d \circ \chi_{i+1})(c \otimes \bar{\alpha} c)$.

As a consequence, 2.1.4 and 2.1.2 follow.

Proof:

$$\chi_i(\bar{\sigma}_L c \otimes \bar{\sigma}_L c) = \chi_i(c \otimes c) + \chi_i(c \otimes \bar{\alpha} c) + \chi_i(\bar{\alpha} c \otimes c) + \chi_i(\bar{\alpha} c \otimes \bar{\alpha} c)$$
$$(\bar{\sigma}_L \circ \chi_i)(c \otimes c) = \chi_i(c \otimes c) + (\bar{\alpha} \circ \chi_i)(c \otimes c) .$$

But $(\bar{\alpha} \circ \chi_i)(c \otimes c) = \chi_i(\bar{\alpha}c \otimes \bar{\alpha}c)$ by 2.2.5, so

$$\begin{aligned}
\chi_i(\bar{\sigma}_L c \otimes \bar{\sigma}_L c) + (\bar{\sigma}_L \circ \chi_i)(c \otimes c) &= \chi_i(c \otimes \bar{\alpha}c + \bar{\alpha}c \otimes c) \\
&= (\chi_i \circ \tau)(c \otimes \bar{\alpha}c) \\
&= (d \circ \chi_{i+1})(c \otimes \bar{\alpha}c) + (\chi_{i+1} \circ d)(c \otimes \bar{\alpha}c) .
\end{aligned}$$

But $d(c \otimes \bar{\alpha}c) = dc \otimes \bar{\alpha}c + c \otimes (\bar{\alpha} \circ d)(c) = 0$, since c was a cocycle. ■

2.3 SOME REMARKS

In this section, we define a version of the transfer map which will be used in Chapter III. We also derive a few formal consequences of Bott's formula, such as the commutativity of the transfer and Steenrod squares in sheaf cohomology for acyclic fixed point set.

2.3.1 DEFINITION

Since the material in Chapter 3 requires a slightly different version of transfer, we briefly list the necessary modifications here.

(1). Basic version.

$$S_L : (\mathscr{B})_{\ell} \overset{\nabla_L}{\rightarrowtail} (\overleftrightarrow{p}\mathscr{B})_{\ell} \overset{\sigma_L}{\twoheadrightarrow} (\mathscr{B})_{\ell}$$ gives rise to the long exact sequence

$$\begin{tikzcd}
 & H^*(Y, \ell, \mathscr{B}) \arrow[dr, "p_L^*"] & \\
H^*(Y, \ell, \mathscr{B}) \arrow[ur, "\mu_L"] & & H^*(X, L, \overleftarrow{p}\mathscr{B}) \arrow[ll, "\Delta_L"]
\end{tikzcd}$$

with connecting morphism μ_L.

The "transfer" Δ_L relates to the Steenrod square as

$$St_i \circ \Delta_L + \Delta_L \circ St_i \;=\; \mu_L \circ St_{i+1} \circ \Delta_L \ ,$$

as was stated in 2.1.5.

(2). $S_0 : (\mathscr{B}) \overset{\nabla_0}{\rightarrowtail} \overleftrightarrow{p}\mathscr{B} \overset{\sigma_0}{\twoheadrightarrow} (\mathscr{B})_{\tilde{\ell}}$ gives rise to the long exact sequence

$$\begin{tikzcd} & H^*(Y, \mathscr{B}) \arrow[dr, "p_0^*"] & \\ H^*(Y, \ell, \mathscr{B}) \arrow[ur, "\mu_0"] & & H^*(X, \overleftarrow{p}\mathscr{B}) \arrow[ll, "\Delta_0"'] \end{tikzcd}$$

with connecting morphism μ_0. We gain the "transfer" Δ_0.

(3). Let ι_ℓ be the natural inclusion $\iota_\ell : H^*(Y, \ell, \mathscr{B}) \to H^*(Y, \mathscr{B})$.
Define $\Delta := \iota_\ell \circ \Delta_0 : H^*(X, \overleftarrow{p}\mathscr{B}) \to H^*(Y, \mathscr{B})$, and call Δ *the absolute transfer.*

In the next statement, we express the commutator of the square

$$\begin{tikzcd} H^i(X, \overleftarrow{p}\mathscr{B}) \arrow[r, "\Delta"] \arrow[d, "Sq^k"'] & H^i(Y, \mathscr{B}) \arrow[d, "Sq^k"] \\ H^{i+k}(X, \overleftarrow{p}\mathscr{B}) \arrow[r, "\Delta"] & H^{i+k}(Y, \mathscr{B}) \end{tikzcd} \qquad (\lozenge)$$

.

2.3.2 LEMMA

$$\Delta_0 \circ Sq^k + Sq^k \circ \Delta_0 = \mu_L \circ Sq^{k-1} \circ \Delta_0 \qquad (*)$$

and, as a consequence,

$$\Delta \circ Sq^k + Sq^k \circ \Delta = \mu_0 \circ Sq^{k-1} \circ \Delta_0 \ . \qquad (**)$$

Proof:

An almost verbatim repetition of the argument in 2.2 gives (*). It hinges on Lemma 2.3.3; we work with the sequence $S_\varnothing$ and its serration resolution. But notice that the representative $\chi_i(c \otimes \overline{\sigma} \circ c)$ in 2.2 is actually in the subsheaf $\mathscr{C}^*(Y, (\overleftrightarrow{p}\mathscr{B})_{\tilde{\prime}})$ of $\mathscr{C}^*(Y, \overleftrightarrow{p}\mathscr{B})$. (**) follows immediately from (*) once we apply $\iota_{\prime}$ to both sides. See also Lemma 1.6.1. ■

2.3.3 LEMMA

The canonical system $\langle \chi_k \mid k = 0, 1, \ldots \rangle$ $\chi_k : \mathscr{C}^*(Y, \overleftrightarrow{p}\mathscr{B}) \otimes \mathscr{C}^*(Y, \overleftrightarrow{p}\mathscr{B}) \to \mathscr{C}^*(Y, \overleftrightarrow{p}\mathscr{B})$ maps the union

$$\mathscr{C}^*(Y, (\overleftrightarrow{p}\mathscr{B})_{\tilde{\prime}}) \otimes \mathscr{C}^*(Y, \overleftrightarrow{p}\mathscr{B}) \cup \mathscr{C}^*(Y, \overleftrightarrow{p}\mathscr{B}) \otimes \mathscr{C}^*(Y, (\overleftrightarrow{p}\mathscr{B})_{\tilde{\prime}})$$

into $\mathscr{C}^*(Y, (\overleftrightarrow{p}\mathscr{B})_{\tilde{\prime}})$.

Proof:

The proof immediately follows as we apply the inductive method of 1.7. This is basically just a consequence of the fact that the product is zero if one of the terms is zero. ■

2.3.4

For the reader's convenience, we summarize 2.3.2 (**) in diagram form:

$$\begin{tikzcd}
 & H^i(X, \overleftarrow{p}\mathscr{B}') \arrow[rr, "\Delta"] \arrow[dl, "\Delta_0"'] \arrow[d, "Sq^k"] & & H^i(Y, \mathscr{B}) \arrow[d, "Sq^k"] & \\
H^i(Y, \ell, \mathscr{B}) \arrow[drr, "Sq^{k-1}"'] & H^{i+k}(X, \overleftarrow{p}\mathscr{B}) \arrow[rr, "\Delta"] & & H^{i+k}(Y, \mathscr{B}) & \\
 & & H^{k+i-1}(Y, \ell, \mathscr{B}) \arrow[ur, "\mu_0"] \arrow[rr, "\mu_L"'] & & H^{k+i}(Y, \ell, \mathscr{B}) \arrow[ul, "\iota_0"']
\end{tikzcd}$$

Now, we return to our original question posed in 2.1.1.

2.3.5 THEOREM

$Sq^k \circ \Delta_L + \Delta_L \circ Sq^k = 0$; i.e., the diagram (*) in 2.1.1 commutes.

Proof:

Suppose that for some $k \geq 0$, $St_k \circ \Delta_L + \Delta_L \circ St_k = 0$. Then for $k+1$ we have $Sq^{k+1} \circ \Delta_L + \Delta_L \circ Sq^{k+1} = \mu_L \circ Sq^k \circ \Delta_L + \mu_L \circ \Delta_L \circ Sq^k$ since $\mu_L \circ \Delta_L = 0$.

For $\mathscr{B} = \mathbb{Z}_2$ we have $Sq^0 = \mathrm{Id}$, and since $Sq^0 \circ \Delta_L + \Delta_L \circ Sq^0 = 2\Delta_L = 0$, the statement follows in this special case.

For a general sheaf $\mathscr{B}$ we do not necessarily have $Sq^0 = \mathrm{Id}$. However, both Sq^0 and Δ_L are induced by a chain map on the canonical flabby resolution. For the former, see 1.8.16. As a consequence, both $Sq^0 \circ \Delta_L$ and $\Delta_L \circ Sq^0$ are induced by a chain map on the canonical flabby resolution. These two maps agree on augmentation level by the commutativity of the diagram on the following page.

$$\begin{array}{ccc} (a, b) & \longrightarrow & (a+b) \\ \downarrow & & \downarrow \\ (a^2, b^2) & \longrightarrow & (a+b)^2 = a^2+b^2 \end{array}$$

The statement now follows by 1.2.8. ■

2.3.6 CONSEQUENCE

The commutator $\Delta \circ Sq^k + Sq^k \circ \Delta$ in 2.3.2 (**) is trivial for fixed-point free actions.

Proof:

If a is fixed-point free, i.e. $L = \mathcal{L} = \varnothing$, then $\Delta = \Delta_L$ and the statement follows from 2.3.5. ■

2.3.7 CONSEQUENCE

Suppose that $H^i(\mathcal{L}, \mathscr{B}|_{\mathcal{L}}) = 0$ for every $i > 0$, i.e. the sheaf $\mathscr{B}|_{\mathcal{L}}$ is acyclic. Then the commutator $\Delta \circ Sq^k + Sq^k \circ \Delta$ is trivial. In particular, if $\mathscr{B} = \mathbb{Z}_2$, then the contractibility of $\mathcal{L}$ implies that the commutator above is trivial.

Proof:

By hypothesis, the map $\iota_{\mathcal{L}} : H^*(Y, \mathcal{L}, \overleftrightarrow{p}\mathscr{B}) \to H^*(Y, \overleftrightarrow{p}\mathscr{B})$ is an isomorphism in positive degrees. By naturality and 2.3.5, it now follows that the commutator is trivial for positive degrees.

In degree zero, $H^0 \cong \Gamma$ and the triviality of $\Delta \circ Sq^0 + Sq^0 \circ \Delta$ immediately follows from the commutativity of the diagram

$$\begin{tikzcd}
(a, b) \arrow[r] \arrow[d] & (a+b) \arrow[d] \\
(a^2, b^2) \arrow[r] & (a+b)^2 = a^2+b^2
\end{tikzcd}$$

as in the proof of 2.3.5. ■

2.3.8

We perform a formal computation here which, in essence, reduces the commutator "to the fixed point set".

Let us start with the formula:

$$P := \Delta \circ Sq^k + Sq^k \circ \Delta = \mu_0 \circ Sq^{k-1} \circ \Delta_0 \quad \text{, i.e., 2.3.2 (**).}$$

Now $Sq^{k-1} \circ \Delta_0 = \Delta_0 \circ Sq^{k-1} + \mu_L \circ Sq^{k-2} \circ \Delta_0$ by 2.3.2 (*). Substituting, we get

$$\begin{aligned} P &= \mu_0 \circ \Delta_0 \circ Sq^{k-1} + \mu_0 \circ \mu_L \circ Sq^{k-2} \circ \Delta_0 \\ &= \mu_0 \circ \mu_L \circ Sq^{k-2} \circ \Delta_0 \ . \end{aligned}$$

Proceeding with $Sq^{k-2} \circ \Delta_0$ as above, we get by induction:

$$\begin{aligned} P &= \mu_0 \sum_{j=1}^{k-1} \mu_L^{j} \circ \Delta_0 \circ Sq^{k-j-1} \\ &= \mu_0 \sum_{j=1}^{k-1} \mu_L^{j-1} \circ (\mu_L \circ \Delta_0) \circ Sq^{k-j-1} \\ &= \mu_0 \sum_{j=1}^{k-1} \mu_L^{j-1} \circ (\delta_{\ell} \circ i_L) \circ Sq^{k-j-1} \\ &= \mu_0 \sum_{j=1}^{k-1} \mu_L^{j-1} \circ \delta_{\ell} \circ Sq^{k-j-1} \circ i_L \qquad \text{by 1.6.1 and 1.6.14.} \end{aligned}$$

Re-indexing and using $\mu_0 = \iota_{\ell} \circ \mu_L$, we get

$$P = \iota_{\ell} \circ \mu_L \sum_{j=0}^{k-2} \mu_L^j \circ \delta_{\ell} \circ Sq^{k-j-2} \circ \underline{i}_L \ .$$

CHAPTER III

3.1 PRELIMINARY DEFINITIONS

In this section, we outline some results concerning the Gysin sequence, excision property, Mayer - Vietoris argument and characteristic class in sheaf - theoretical context.

3.1.1 DEFINITION

Let X be a paracompact topological space and let F be a closed subspace. Let $\mathscr{A}$ be a sheaf of Z_2-algebras. Then, as it is well known, we can define the module structures:

$$\cup : H^*(X, Z_2) \otimes H^*(X, \mathscr{A}) \to H^*(X, \mathscr{A})$$

$$H^*(F, Z_2) \otimes H^*(F, \mathscr{A}|_F) \to H^*(F, \mathscr{A}|_F)$$

and $$H^*(X, Z_2) \otimes H^*(X, F, \mathscr{A}) \to H^*(X, F, \mathscr{A})$$

One possible definition can be obtained by generalizing the machinery in 1.1.7 to more complicated pairings; indeed, when $\mathscr{A} = Z_2$, $\cup$ is induced by the map χ_0 . For other possible definitions, such as, (for example) in terms of the A^n - representation, see **[Bredon 1]**. The structure $\cup$ has several desirable features. First of all, this structure is natural with respect to cohomomorphism of Z_2-algebras; in particular, it is compatible with the restriction cohomomorphism. One of the more elementary properties is the following compatibility.

Let

$$R_F : \quad (\mathscr{A})_{\bar{F}} \overset{\iota_F}{\rightarrowtail} \mathscr{A} \overset{\rho_F}{\twoheadrightarrow} (\mathscr{A})_F$$

be the relative sequence on X . Let δ_F be the connecting morphism

$$\delta_F: \quad H^*(F, \mathscr{A}|_F) \to H^*(X, F, \mathscr{A})$$

Let $[c] \in H^*(F, \mathscr{A}|_F)$ and $[b] \in H^*(X, \mathbb{Z}_2)$. Then we have the following.

3.1.2 LEMMA

$$\delta_F(i_F[b] \cup [c]) = [b] \cup \delta_F[c]$$

Proof:

Easy diagram chasing and the fact that $\cup$ is induced by a map which is a derivation on chain level gives the lemma. ■

3.1.3

Next we establish an excision property we will use later. Considerably stronger excision/Mayer-Vietoris properties exist in sheaf theory; however, the version below is sufficient for our purposes, and we will confine ourselves to this weaker statement. Let F, N ; $F \subset N \subset X$; be closed subspaces with N being a regular closed neighborhood of F in X ; i.e., $\operatorname{int}(N) \supset F$.

Suppose $X\setminus F$ is a paracompact subspace. Notice that the paracompactness of $X\setminus F$ holds automatically if X is completely paracompact (e.g., X metrizable). Let $\mathscr{A}$ be a sheaf on X. We have the natural cohomomorphism $i_{X\setminus F}: (\mathscr{A})_{\tilde{N}} \to (\mathscr{A}|_{X\setminus F})_{\widetilde{N\setminus F}}$.

3.1.4 LEMMA (EXCISION)

Under the conditions above, $i_{X\setminus F}: H^*(X, N, \mathscr{A}) \to H^*(X\setminus F, N\setminus F, \mathscr{A})$ is an isomorphism.

Proof: See **[Bredon 1]**. ■

3.1.5 MAYER - VIETORIS ARGUMENTS

Mayer - Vietoris sequences can be constructed in sheaf theory as was shown in **[Bredon 1]**. In fact, granted a certain excision property, the Eilenberg - Steenrod axioms give rise to such sequences; see also **[Eilenberg - Steenrod]**.

Let X_1 and X_2 be two paracompact subspaces such that

(1). $X_1 \cup X_2 = X$

(2). X_1 and X_2 are excisive, i.e. the natural cohomomorphism

$$i_{X_1} : H^*(X, X_2, \mathscr{A}) \to H^*(X_1, X_1 \cap X_2, \mathscr{A}|_{X_1})$$

induces an isomorphism for every sheaf $\mathscr{A}$ on X.

Then there is a long exact sequence of the form

M:
$$\begin{tikzcd}[column sep=tiny]
 & H^*(X, \mathscr{A}) \arrow[dr, "i_{X_1} \oplus i_{X_2}"] & \\
H^*(X_1 \cap X_2, \mathscr{A}|_{X_1 \cap X_2}) \arrow[ur, "D"] & & H^*(X_1, \mathscr{A}|_{X_1}) \oplus H^*(X_2, \mathscr{A}|_{X_2}) \arrow[ll, "i^{X_1}_{X_1 \cap X_2} \oplus i^{X_2}_{X_1 \cap X_2}"]
\end{tikzcd}$$

with connecting morphism D.

The connecting morphism D can be computed as the composition

$$\begin{tikzcd}
H^*(X_1 \cap X_2, \mathscr{A}|_{X_1 \cap X_2}) \arrow[r, "\delta^{X_2}_{X_1 \cap X_2}"] \arrow[d, "D"'] & H^*(X_1, X_1 \cap X_2, \mathscr{A}|_{X_1}) \arrow[d, shift left=2, "(i_{X_1})^{-1}"] \\
H^*(X, \mathscr{A}) & H^*(X, X_2, \mathscr{A}) \arrow[l, "\iota_{X_2}"] \arrow[u, shift left=2, "i_{X_1}"] \arrow[u, phantom, "\equiv"]
\end{tikzcd}$$

i.e. $D = \iota_{X_2} \circ (\underline{i}_{X_1})^{-1} \circ \delta^{X_2}_{X_1 \cap X_2}$. As a consequence, the exact sequence M is natural with respect to cohomomorphism of sheaves covering maps of pairs.

Notice that by 3.1.4 the triad (X, X\F, N) is excisive. Consequently, we define the connecting morphism D_N of the corresponding Mayer - Vietoris sequence, which in this case is of the form

$$\begin{tikzcd} H^*(N\backslash F, \mathscr{A}|_{N\backslash F}) \arrow[r, "\delta^{X\backslash F}_{N\backslash F}"] \arrow[d, "D_N"'] & H^*(X\backslash F, N\backslash F, \mathscr{A}|_{X\backslash F}) \arrow[d, "(\underline{i}_{N\backslash F})^{-1}"] \\ H^*(X, \mathscr{A}) & H^*(X, N, \mathscr{A}) \arrow[l, "\iota_N"] \end{tikzcd} ..$$

3.1.6 GYSIN SEQUENCE

The third tool we are going to use is the characterization of the Smith sequence for fixed-point free involutions. Suppose X is a paracompact Hausdorff space and a is a fixed-point free involution. Let Y be the quotient. Then if $\mathscr{B}$ is some sheaf of Z_2-modules on Y , the corresponding Smith sequence is of the form

$$S_0 = G_0 : \quad \begin{tikzcd} & H^*(Y, \mathscr{B}) \arrow[dr, "p_0^*"] & \\ H^*(Y, \mathscr{B}) \arrow[ur, "\mu_0"] & & H^*(X, \overleftarrow{p}\mathscr{B}) \arrow[ll, "\Delta_0"] \end{tikzcd}$$

and μ_0 is given by $\mu_0 [c] = \Omega \cup [c]$ where $\Omega \in H^1(Y, Z_2)$, a fixed element called the *characteristic class* (S_0 is called the *Gysin sequence*). Since the proof requires different machinery which we will not be using in our further discussions, it was decided to relegate this proof to Appendix A. A more general framework in which to construct Gysin sequences can be found in **[Bredon 1]**. The fact that the Smith sequence and the Gysin sequence correspond for double covers was proved in **[Bredon 2]** for constant sheaf coefficients.

Changing the sheaf as in 1.5.19, we can derive the relative version as

$$S_N = G_N : \quad \begin{tikzcd} & H^*(Y, \mathscr{n}, \mathscr{B}) \arrow[dr, "p_N^*"] & \\ H^*(Y, \mathscr{n}, \mathscr{B}) \arrow[ur, "\Omega \cup \cdot"] & & H^*(X, N, \overleftarrow{p}\mathscr{B}) \arrow[ll, "\Delta_N"] \end{tikzcd}$$

S_N and S_0, being Smith sequences in a special case, have the naturality properties stated in 1.5.19. Not surprisingly, the characteristic class Ω is itself natural with respect to restrictions; i.e. if $X \subset X'$ as in 1.5.10, then $\Omega = \underline{i}_Y\Omega'$.

3.2 TWO NATURAL TRANSFORMATIONS

In this section, we define two natural transformations and prove their equivalence. This fact will allow us to express the commutator between Steenrod squares and transfer in a form that is reminiscent of the classical duality theorems of algebraic topology.

3.2.1 DEFINITION

Let (X, a) be a space with involution and $\mathscr{n}$ be a regular closed neighborhood of ℓ. Let j be some non-negative integer. Define $P_j^{\mathscr{n}} := H^*(\mathscr{n}, \mathscr{B}|_{\mathscr{n}}) \to H^*(Y, \mathscr{B})$ as the composition:

$$\begin{tikzcd}
H^*(\mathfrak{n}, \mathscr{S}|_{\mathfrak{n}}) \arrow[r, "\delta_*"] \arrow[d, "P_j^*"'] & H^*(Y, \mathfrak{n}, \mathscr{S}) \arrow[d, "(\mu_N)^j"] \\
H^*(Y, \mathscr{S}) & H^*(Y, \mathfrak{n}, \mathscr{S}) \arrow[l, "\iota_*"]
\end{tikzcd}$$

where, of course, $(\mu_N)^j$ stands for $\mu_N \circ \mu_N \circ \cdots \circ \mu_N$ (j times). By the naturality property of the maps involved, we have the following lemma which the reader may easily verify.

3.2.2 LEMMA

Let $\mathfrak{n} \subset \mathfrak{n}'$ be regular neighborhoods as in 3.2.1. Then

$$\begin{tikzcd}
H^*(\mathfrak{n}', \mathscr{S}|_{\mathfrak{n}'}) \arrow[d, "\underline{i}_{\mathfrak{n}}^{\mathfrak{n}'}"'] \arrow[dr, "P_j^{\mathfrak{n}'}"] & \\
H^*(\mathfrak{n}, \mathscr{S}|_{\mathfrak{n}}) \arrow[r, "P_j^{\mathfrak{n}}"'] & H^*(Y, \mathscr{S})
\end{tikzcd}$$

commutes. ∎

3.2.3 DEFINITION

Let $\mathcal{N} := \{ \mathfrak{n} \mid \mathfrak{n}$ is a closed regular neighborhood of $\ell \}$.

Evidently, $\mathcal{N}$ is a directed set with respect to containment. On cohomology level, we get the corresponding direct system of groups $\{H^*(\mathfrak{n}, \mathscr{S}|_{\mathfrak{n}}), \underline{i} \mid \mathfrak{n} \in \mathcal{N}\}$. By 3.2.2, P_j^* descends to a map $P_j = \varinjlim_{\mathfrak{n} \in \mathcal{N}} P_j^*$; $P_j = \varinjlim_{\mathfrak{n} \in \mathcal{N}} \{H^*(\mathfrak{n}, \mathscr{S}|_{\mathfrak{n}})\} \to H^*(Y, \mathscr{S})$.

3.2.4 DEFINITION

Next, we define another functor Q_j. Its definition is reminiscent of the construction of other Poincaré or Alexander-Spanier - type maps. Note that the orientation class gets replaced with the characteristic class in our case.

Let Ω_N be the characteristic class of the fixed-point free involution $(N\backslash L, a|_{N\backslash L})$. The quotient space is, of course, $\mathscr{n}\backslash\mathscr{l}$. Define $Q_j^*: H^*(\mathscr{n}, \mathscr{S}|_{\mathscr{n}}) \to H^*(Y, \mathscr{S})$ by the composition:

$$\begin{tikzcd}
H^*(\mathscr{n}, \mathscr{S}|_{\mathscr{n}}) \arrow[r, "i^*_{\mathscr{n}\backslash\mathscr{l}}"] \arrow[d, "Q_j^*"'] & H^*(\mathscr{n}\backslash\mathscr{l}, \mathscr{S}|_{\mathscr{n}\backslash\mathscr{l}}) \arrow[d, "(\mu_0^{*\backslash\mathscr{l}})^j = \Omega_N^j \cup \cdot"] \\
H^*(Y, \mathscr{S}) & H^*(\mathscr{n}\backslash\mathscr{l}, \mathscr{S}|_{\mathscr{n}\backslash\mathscr{l}}) \arrow[l, "D_{\mathscr{n}\backslash\mathscr{l}}"]
\end{tikzcd},$$

where $D_{\mathscr{n}\backslash\mathscr{l}}$ is the boundary operator of the Mayer - Vietoris sequence for the triad $(Y, Y\backslash\mathscr{l}, \mathscr{n})$ and $\Omega_N^j = \Omega_N \cup \Omega_N \cup \ldots \cup \Omega_N$ (j times). Again, by naturality, we can conclude the following as an obvious fact.

3.2.5 LEMMA

Let $\mathscr{n} \subset \mathscr{n}'$ be regular neighborhoods as in 3.2.1. Then

$$\begin{tikzcd}
H^*(\mathscr{n}', \mathscr{S}|_{\mathscr{n}'}) \arrow[d, "i^{*'}_{*}"'] \arrow[dr, "Q_j^{*'}"] & \\
H^*(\mathscr{n}, \mathscr{S}|_{\mathscr{n}}) \arrow[r, "Q_j^*"'] & H^*(Y, \mathscr{S})
\end{tikzcd}$$

commutes. ■

3.2.6 DEFINITION

Again we define $Q_j = \varinjlim_{\mathcal{N} \in \mathcal{N}} Q_j^{\mathcal{N}}$; $Q_j = \varinjlim_{\mathcal{N} \in \mathcal{N}} \{H^*(\mathcal{N}, \mathcal{B}|_{\mathcal{N}})\} \to H^*(Y, \mathcal{B})$.

Next we prove that, in fact, we defined the *same* natural transformation of functors. Indeed, we have:

3.2.7 LEMMA

Suppose $X \setminus \mathcal{L}$ is paracompact. Then $P_j^{\mathcal{N}} = Q_j^{\mathcal{N}}$ for every regular neighborhood $\mathcal{N}$ as in 3.2.1. As a consequence, $P_j = Q_j$.

<u>Proof</u>:

Fix $\mathcal{N}$.

(1). $i_{Y\setminus\mathcal{L}}$ induces a map between the Smith sequences corresponding to (X, N) and $(X\setminus L, N\setminus L)$; see 1.5.19. In particular, the following diagram commutes.

$$\begin{tikzcd}
H^*(Y, \mathcal{N}, \mathcal{B}) \arrow[r, "\mu_N"] \arrow[d, "i_{Y\setminus\mathcal{L}}"'] & H^*(Y, \mathcal{N}, \mathcal{B}) \arrow[d, "i_{Y\setminus\mathcal{L}}"] \\
H^*(Y\setminus\mathcal{L}, \mathcal{N}\setminus\mathcal{L}, \mathcal{B}|_{Y\setminus\mathcal{L}}) \arrow[r, "\mu_{N\setminus L}"'] & H^*(Y\setminus\mathcal{L}, \mathcal{N}\setminus\mathcal{L}, \mathcal{B}|_{Y\setminus\mathcal{L}})
\end{tikzcd}$$

Since the action on $X\setminus L$ is free, the second sequence is in fact a Gysin sequence. Consequently, for any $[a] \in H^*(Y, \mathcal{N}, \mathcal{B})$:

$$\Omega^j_{Y\setminus\mathcal{L}} \cup (i_{Y\setminus\mathcal{L}}[a]) \;=\; \mu^j_{N\setminus\mathcal{L}} \circ i_{Y\setminus\mathcal{L}}[a] \;=\; i_{Y\setminus\mathcal{L}} \circ \mu^j_N[a]$$

(2). Let $[c] \in H^*(\mathcal{N}, \mathcal{B}|_{\mathcal{N}})$ be a cohomology class. Apply the previous equality with $[a] := \delta_{\mathcal{N}}[c] \in H^*(Y, \mathcal{N}, \mathcal{B})$ to get the following.

$$\Omega^{j}_{Y\backslash\ell} \cup (\underline{i}_{Y\backslash\ell} \circ \delta_{\mathcal{N}}[c]) \;=\; \underline{i}_{Y\backslash\ell} \circ \mu^{j}_{N} \circ \delta_{\mathcal{N}}[c]$$

By naturality, we have the commutative diagram

$$\begin{tikzcd}
H^*(\mathcal{N}, \mathcal{B}|_{\mathcal{N}}) \arrow[r, "\delta_{\mathcal{N}}"] \arrow[d, "\underline{i}_{\mathcal{N}\backslash\ell}"'] & H^*(Y, \mathcal{N}, \mathcal{B}) \arrow[d, "\underline{i}_{Y\backslash\ell}", "\cong"'] \\
H^*(\mathcal{N}\backslash\ell, \mathcal{B}|_{\mathcal{N}\backslash\ell}) \arrow[r, "\delta_{\mathcal{N}\backslash\ell}"'] & H^*(Y\backslash\ell, \mathcal{N}\backslash\ell, \mathcal{B}|_{Y\backslash\ell})
\end{tikzcd}$$

with the second vertical arrow being an isomorphism by 3.1.4. Consequently, we have $\Omega^{j}_{Y\backslash\ell} \cup (\delta_{\mathcal{N}\backslash\ell} \circ \underline{i}_{\mathcal{N}\backslash\ell}[c]) \;=\; \underline{i}_{Y\backslash\ell} \circ \mu^{j}_{N} \circ \delta_{\mathcal{N}}[c]$.

(3). By Lemma 3.1.2, $\delta_{\mathcal{N}\backslash\ell}\,((\underline{i}_{\mathcal{N}\backslash\ell}\,\Omega_{Y\backslash\ell})^{j} \cup (\underline{i}_{\mathcal{N}\backslash\ell}[c])) \;=\; \underline{i}_{Y\backslash\ell} \circ \mu^{j}_{N} \circ \delta_{\mathcal{N}}[c]$ and, by the naturality of the characteristic class, $\delta_{\mathcal{N}\backslash\ell}(\Omega^{j}_{\mathcal{N}\backslash\ell} \cup (\underline{i}_{\mathcal{N}\backslash\ell}[c])) \;=\; \underline{i}_{Y\backslash\ell} \circ \mu^{j}_{N} \circ \delta_{\mathcal{N}}[c]$.

(4). Since $\underline{i}_{Y\backslash\ell}$ is an isomorphism, as we have already observed, we write

$$(\underline{i}_{Y\backslash\ell})^{-1} \circ \delta_{\mathcal{N}\backslash\ell}(\Omega^{j}_{\mathcal{N}\backslash\ell} \cup \underline{i}_{\mathcal{N}\backslash\ell}[c]) = \mu^{j}_{N} \circ \delta_{\mathcal{N}}[c] \;.$$

Applying $\iota_{\mathcal{N}}$ on both sides,

$$\iota_{\mathcal{N}} \circ (\underline{i}_{Y\backslash\ell})^{-1} \circ \delta_{\mathcal{N}\backslash\ell}(\Omega^{j}_{\mathcal{N}\backslash\ell} \cup \underline{i}_{\mathcal{N}\backslash\ell}[c]) = \iota_{\mathcal{N}} \circ \mu^{j}_{N} \circ \delta_{\mathcal{N}}[c] \quad \text{in } H^*(Y, \mathcal{B}) .$$

(5). By 3.1.5, $\iota_{\mathcal{N}} \circ (\underline{i}_{Y\backslash\ell})^{-1} \circ \delta_{\mathcal{N}\backslash\ell} = D_{\mathcal{N}\backslash\ell}$. Now by definition, the last equality is precisely $Q^{\mathcal{N}}_{j}[c] \;=\; P^{\mathcal{N}}_{j}[c]$. ■

3.3 ANOTHER CHARACTERIZATION OF THE COMMUTATOR

In this section we give our generalization of Kubelka's formula for sheaf coefficients. If we take $\mathscr{B}$ to be the constant sheaf $\mathbb{Z}_2$, then we can easily derive Kubelka's original formula without imposing overly restrictive simplicial conditions on the space or the involution.

3.3.1

Since $\mathscr{n} \supset \ell$ for any $\mathscr{n} \in \mathscr{N}$, the inclusion naturally induces a map

$$\underline{i} : \varinjlim_{\mathscr{n} \in \mathscr{N}} \{H^*(\mathscr{n}, \mathscr{B}|_{\mathscr{n}})\} \to H^*(\ell, \mathscr{B}|_{\ell}) \; .$$

The fixed point set ℓ is closed; hence by **[Bredon 1]** we have the following well-known fact.

3.3.2 FACT

The map $\underline{i}$ is an isomorphism. ■

3.3.3

By 3.2.7, P_j and Q_j respectively induce the maps

$$P_j \circ (\underline{i})^{-1} : H^*(\ell, \mathscr{B}|_{\ell}) \to H^*(Y, \mathscr{B}) \quad \text{and} \quad Q_j \circ (\underline{i})^{-1} : H^*(\ell, \mathscr{B}|_{\ell}) \to H^*(Y, \mathscr{B}) \, .$$

By 3.2.6, we have that $P_j \circ (\underline{i})^{-1} = Q_j \circ (\underline{i})^{-1}$. Then by 3.2.2, we can conclude that $P_j \circ (\underline{i})^{-1}[c] = \iota_{\ell} \circ \mu_L^j \circ \delta_{\ell}[c]$ for any $[c] \in H^*(\ell, \mathscr{B}|_{\ell})$. In fact, we can put this fact together with the formula in 2.3.8 to express the commutator $P = \Delta \circ Sq^k + Sq^k \circ \Delta$.

By 2.3.8,

$$P = \iota_{\prime} \circ \mu_L \left(\sum_{j=0}^{k-2} \mu_L^j \circ \delta_{\prime} \circ Sq^{k-j-2} \circ \underline{i}_{\prime} \right)$$

$$= \sum_{j=0}^{k-2} \iota_{\prime} \circ \mu_L^{j+1} \circ \delta \circ Sq^{k-j-2} \circ \underline{i}_{\prime}$$

$$= \sum_{j=1}^{k-1} P_j \circ (\underline{i})^{-1} \circ Sq^{k-j-1} \circ \underline{i}_{\prime} \quad .$$

Thus we have derived the following generalization of Kubelka's theorem (see **[Kubelka]** or section 1.1.4).

3.3.4 THEOREM

Suppose that X is a paracompact Hausdorff space, a is an involution on X, and X\L is paracompact where L stands for the fixed point set of a. Let $\mathscr{B}$ be a sheaf of $\mathbb{Z}_2$ - algebras on the quotient space Y. Then the commutator P between the kth Steenrod square and the transfer map can be expressed as:

$$P = \sum_{j=1}^{k-1} Q_j \circ (\underline{i})^{-1} \circ Sq^{k-j-1} \circ \underline{i}_{\prime} \quad . \quad \blacksquare$$

3.3.5 OBSERVATION

$Q_j \circ (\underline{i})^{-1}$ reflects how the fixed point set is actually imbedded into X and Y, while the remaining part of the formula reflects the intrinsic geometry of the fixed point set.

3.3.6

Generally speaking, it is very hard to give any tangible representation for the map $Q_j \circ (\underline{i})^{-1} : H^*(\mathcal{L}, \mathscr{B}|_{\mathcal{L}}) \to H^*(Y, \mathscr{B})$ since such a representation would require a right inverse to the restriction; i.e., an "extension" map. However, if we have constant sheaf coefficients, we can easily give such an extension provided the fixed point set $\mathcal{L}$ is a neighborhood retract.

Suppose that $\mathscr{B}$ is a constant sheaf and $r : \mathcal{N} \to \mathcal{L}$ is a retraction; that is, a continuous map with $r|_{\mathcal{L}} = \mathrm{Id}_{\mathcal{L}}$. Then r is covered by a unique cohomomorphism, $\underline{r} : \mathscr{B}|_{\mathcal{L}} \to \mathscr{B}|_{\mathcal{N}}$ and, evidently, $\underline{i}_{\mathcal{L}} \circ \underline{r}$ is the identity homomorphism of $\mathscr{B}|_{\mathcal{L}}$.

Let $\dot{r} := r|_{\mathcal{N}\setminus\mathcal{L}}$.

Let $\underline{\dot{r}}$ be the unique cohomomorphism of constant sheaves covering $\dot{r}$. Then, by definition, for $[a] \in H^*(\mathcal{L}, \mathscr{B}|_{\mathcal{L}})$ we have $(Q_j \circ (\underline{i})^{-1})[a] = D_{\mathcal{N}\setminus\mathcal{L}}\,(\Omega_N^j \cup (\,\underline{\dot{r}}\,[a]\,))$, and 3.3.4 takes the more familiar form as in **[Kubelka]**:

3.3.7 THEOREM

$$P([c]) \;=\; D_{\mathcal{N}\setminus\mathcal{L}}\Big(\sum_{j=1}^{k-1} \Omega_N^j \cup ((\underline{\dot{r}} \circ Sq^{k-j-1} \circ \underline{i}_L)[c])\Big)$$

$$=\; D_{\mathcal{N}\setminus\mathcal{L}}\Big(\sum_{j=0}^{k-1} \Omega_N^{k-j-1} \cup ((\underline{\dot{r}} \circ Sq^{j} \circ \underline{i}_L)[c])\Big)$$

where $[c] \in H^*(X, \overleftarrow{p}\mathscr{B})$. ■

CONCLUDING REMARKS

(1). Throughout this thesis we worked under the general assumption that X was a paracompact space. While this supposition helped us to avoid the repetitious burden of stating paracompactness as a condition in many of our statements, it should be pointed out that a significant portion of the thesis remains true without the assumption of paracompactness. The only chapter where paracompactness seems to be more or less indispensible is Chapter III.

(2). Also, throughout this thesis we worked under a specific support system, namely the system consisting of all closed sets of the space in point. Nearly all of the theorems could have been stated for more general support systems. One particularly interesting case is the system consisting of all compact subsets. Using this system, we believe it is possible to give a generalization of both Bott's and Kubelka's results for cohomology with compact support.

(3). We now strongly suspect that most results in [Bott] can be generalized in the manner of this thesis. Such a generalization might yield an alternative construction of Steenrod squares via Smith sequences and symmetric products, at least for $\mathbb{Z}_2$ - algebra coefficient sheaves.

(4). Another interesting problem encountered while working on this thesis concerns the description of Steenrod squares in terms of the Godement representation described in sections 1.3 and 1.4. It may be possible for the interested reader to find a relatively simple description of the homotopy system $\langle \chi_k \mid k = 0, 1, \ldots \rangle$ in the setting of section 1.4. For cup products (i.e. St_0) such a description has been carried out in **[Bredon 1]** and **[Godement]**. A good starting point for general Steenrod squares may be a close investigation of the methods in **[Steenrod 1]**. The essential ideas discussed in Steenrod's original paper should also be able to be presented in sheaf - theoretical context.

A significantly harder task would be the description of Steenrod p - powers in terms of the Godement representation. We found no indication that such a description has ever been attempted, either in sheaf - theoretical or Alexander - Spanier cohomological context.

(5). Kubelka describes a way to compute the commutator between Steenrod squares and transfer in the more general setting when a is a homeomorphism of period 2^n (see **[Kubelka]**). Since his method is basically an iterated application of the formula in 3.3, his results can be repeated in our sheaf - theoretical context just as well.

(6). Sheaf theory has strong ties with obstruction theory via local coefficient systems associated to fibrations. Local systems are also closely connected with equivariant cohomology theories. It may be interesting to investigate whether or not our commutator formula has any relevance in this context. See **[Whitehead]** and **[Spanier]** for general information.

(7). Another unsolved question that should be stated here concerns the generalization of the computations in this present thesis to Steenrod p - powers. Suppose G is acting on X and p is a prime. Can we give an explicit formula for the commutator between Steenrod p - powers and the transfer arising from the G - action? Is there a generalization for sheaf coefficients? As is demonstrated in Appendix B, it is logical to restrict ourselves to the case $G = \mathbb{Z}_p$. As far as we know, the only result along these lines is **[Liao]** . Here the author carries out some computations for symmetric products of spheres.

We ourselves are not entirely convinced that the commutator above permits a reasonably simple expression; in fact, the tendency is to think that it does not. However, if such an expression were possible for finite simplicial complexes and regular actions, we believe it would be possible to generalize the result to twisted coefficient systems and more general spaces via the machinery developed in this thesis.

Appendix A

In this appendix, we prove that the connecting morphism of the Smith sequence corresponding to a double cover $p: X \to Y$ is given as multiplication by the characteristic class. We use the Čech representation of sheaf cohomology as in [Spanier], p. 327.

We call a cover "good" if the intersection of any number of cover elements is connected. We suppose that the good covers are cofinal in the set of all locally finite covers. It will be evident that this supposition is not really essential in our computation; however, it greatly simplifies the notation. Otherwise, we would be obliged to index the components, which we want to avoid below.

The Smith sequence is now represented as the direct limit of short exact sequences

$$C^*(\mathcal{U}, \mathcal{B}) \overset{i}{\rightarrowtail} C^*(\mathcal{U}, \overleftrightarrow{p}\mathcal{B}) \overset{\sigma}{\twoheadrightarrow} C^*(\mathcal{U}, \mathcal{B})$$

where the limit is taken over the set $\{ \mathcal{U} \mid \mathcal{U} \text{ is a good cover of } Y \}$.

Let $[c] \in H^{k-1}(Y, \mathcal{B})$ be a cohomology class given by the cocycle $c \in C^{k-1}(\mathcal{U}, \mathcal{B})$ on some cover $\mathcal{U}$. Suppose $\mathcal{U}$ is fine enough to trivialize the double cover $p: X \to Y$. Fix an ordering on $\mathcal{U}$. Since $\mathcal{U}$ is a trivializing cover,

$$\Gamma(U, \overleftrightarrow{p}\mathcal{B}) \cong \mathcal{B}|_U \oplus \mathcal{B}|_U \quad \text{for every } U \in \mathcal{U}.$$

Fix such an isomorphism.

Next, trivialize $\overleftrightarrow{p}\mathcal{B}|_{U_0 \cap \cdots \cap U_n}$ for any system of intersections in the same sense that U_0 trivialized $\overleftrightarrow{p}\mathcal{B}$; here we have used the convention that intersections are written in decreasing order (i.e., $U_0 > U_1 > \dots > U_n$). We use this convention throughout our argument. In fact, we defined an isomorphism

$$C^*(\mathcal{U}, \overset{\leftrightarrow}{p}\mathcal{B}) \cong C^*(\mathcal{U}, \mathcal{B}) \oplus C^*(\mathcal{U}, \mathcal{B})$$

of Abelian groups. In this sense,we will talk about "left" and "right" components of an element in $C^*(\mathcal{U}, \overset{\leftrightarrow}{p}\mathcal{B})$. Notice that the differentiation d does not respect the splitting.

The characteristic class $\Omega \in H^1(Y, \mathbb{Z}_2)$ is given by the cocycle $\Omega \in C^1(\mathcal{U}, \mathbb{Z}_2)$ that associates 1 to $U_0 \cap U_1$ if $\overset{\leftrightarrow}{p}\mathcal{B}$ is trivialized in opposite sense, and 0 otherwise.

Next, we define the polarization operator:

$$K: \quad C^{k-1}(\mathcal{U}, \mathcal{B}) \longrightarrow C^{k-1}(\mathcal{U}, \overset{\leftrightarrow}{p}\mathcal{B}) \cong C^{k-1}(\mathcal{U}, \mathcal{B}) \oplus C^{k-1}(\mathcal{U}, \mathcal{B})$$

Let $s \in \Gamma(U_0 \cap U_1 \cap \dots \cap U_{k-1}, \mathcal{B})$ be a section.

Define $Ks \in \Gamma(U_0 \cap U_1 \cap \dots \cap U_{k-1}, \overset{\leftrightarrow}{p}\mathcal{B})$ by $\begin{cases} s \oplus o & \text{if } \Omega(U_0, U_1) = 1 \\ o \oplus s & \text{if } \Omega(U_0, U_1) = 0 \end{cases}$.

I.e. K "sweeps" the section to the "left" or to the "right" with respect to the given trivializations. Evidently $\sigma \circ K = \mathrm{Id}$ on $C^{k-1}(\mathcal{U}, \mathcal{B})$. Consequently, the value of the connecting map in the Smith sequence is given by $(d \circ K)c \in C^k(\mathcal{U}, \mathcal{B})$. We shall compute this cocycle explicitly. We need the following auxiliary operators:

(1). Let

$$T(U_0, U_1): \quad \Gamma(U_1 \cap U_2 \cap \dots \cap U_k, \mathcal{B}) \oplus \Gamma(U_1 \cap U_2 \cap \dots \cap U_k, \mathcal{B}) \longrightarrow \Gamma(U_0 \cap U_1 \cap \dots \cap U_k, \mathcal{B}) \oplus \Gamma(U_0 \cap U_1 \cap \dots \cap U_k, \mathcal{B})$$

be the restriction $\Gamma(U_1 \cap U_2 \cap \dots \cap U_k, \overset{\leftrightarrow}{p}\mathcal{B}) \longrightarrow \Gamma(U_0 \cap U_1 \cap \dots \cap U_k, \overset{\leftrightarrow}{p}\mathcal{B})$

represented in terms of the trivializations. In other words, $T(U_0, U_1)$ is either the direct sum of restrictions or the direct sum of restrictions composed with the natural "switch", depending on $\Omega(U_0, U_1)$. The first case corresponds to $\Omega(U_0, U_1) = 0$, the second to $\Omega(U_0, U_1) = 1$.

(2). Suppose $U_0 \cap U_1 \cap \ldots \cap U_k \neq \varnothing$. Define

$$K(U_i, U_j): \Gamma(U_0 \cap \ldots \cap \hat{U}_i \cap \ldots \cap U_k, \mathscr{B}) \oplus \Gamma(U_0 \cap \ldots \cap \hat{U}_j \cap \ldots \cap U_k, \mathscr{B}) \rightarrow \Gamma(U_0 \cap U_1 \cap \ldots \cap U_k, \overleftrightarrow{p}\mathscr{B})$$

to be the restriction to $U_0 \cap U_1 \cap \ldots \cap U_k$ composed with the polarization in degree $(k+1)$ depending on $\Omega(U_i, U_j)$. In other words, we restrict the section and then write it to the "left" or to the "right" with respect to the trivialization

$$\Gamma(U_0 \cap U_1 \cap \ldots \cap U_k, \overleftrightarrow{p}\mathscr{B}) \cong \Gamma(U_0 \cap U_1 \cap \ldots \cap U_k, \mathscr{B}) \oplus \Gamma(U_0 \cap U_1 \cap \ldots \cap U_k, \mathscr{B})$$

depending on $\Omega(U_i, U_j)$.

Next, we compute $(d \circ K)c$.

$$\begin{aligned}
((d \circ K)c)(U_0, U_1, U_2, \ldots, U_k) = \\
&\Big\{\sum_{i=2}^{k} K[\,c\,(U_0, U_1, \ldots, \hat{U}_i, \ldots, U_k)] + K[\,c\,(U_0, U_2, \ldots, U_k)]\Big\}\Big|_{U_0 \cap U_1 \cap \cdots \cap U_k} \\
&\qquad + T(U_0, U_1) \circ K[\,c\,(U_1, U_2, \ldots, U_k)] \\
= \; & T(U_0, U_1) \circ K[\,c\,(U_1, U_2, \ldots, U_k)] + K[\,c\,(U_0, U_2, \ldots, U_k)]\Big|_{U_0 \cap U_1 \cap \cdots \cap U_k} \\
&\qquad + K(U_0, U_1)[\,c\,(U_1, U_2, \ldots, U_k)] + K(U_0, U_1)[\,c\,(U_0, U_2, \ldots, U_k)]
\end{aligned}$$

where the last equality follows from the fact that c was a cocycle. Since Ω is a cocycle, we have the following four cases:

(i). $\Omega(U_0, U_1) = 0$

$\Omega(U_1, U_2) = 0$

(consequently, $\Omega(U_0, U_2) = 0$)

In this case, $((d \circ K)c)(U_0, U_1, U_2, \ldots, U_k) = 0$.

(ii). $\Omega(U_0, U_1) = 0$

$\Omega(U_1, U_2) = 1$

(consequently, $\Omega(U_0, U_2) = 1$)

In this case, $((d \circ K)c)(U_0, U_1, U_2, \ldots, U_k) =$

$$c(U_1, U_2, \ldots, U_k)|_{U_0 \cap U_1 \cap \cdots \cap U_k} + c(U_0, U_2, \ldots, U_k)|_{U_0 \cap U_1 \cap \cdots \cap U_k}$$

(iii). $\Omega(U_0, U_1) = 1$

$\Omega(U_1, U_2) = 0$

(consequently, $\Omega(U_0, U_2) = 1$)

In this case, $((d \circ K)c)(U_0, U_1, U_2, \ldots, U_k) = 0$.

(iv). $\Omega(U_0, U_1) = 1$

$\Omega(U_1, U_2) = 1$

(consequently, $\Omega(U_0, U_2) = 0$)

In this case, $((d \circ K)c)(U_0, U_1, U_2, \ldots, U_k) =$

$$c(U_1, U_2, \ldots, U_k)|_{U_0 \cap U_1 \cap \cdots \cap U_k} + c(U_0, U_2, \ldots, U_k)|_{U_0 \cap U_1 \cap \cdots \cap U_k}$$

Next, let $b \in C^{k-1}(\mathcal{U}, \mathcal{B})$ be the cochain defined by

$b(U_0, U_1, \ldots, U_{k-1}) = c(U_0, U_1, \ldots, U_{k-1})$ if $\Omega(U_0, U_1) = 1$;

$b(U_0, U_1, \ldots, U_{k-1}) = 0$ if $\Omega(U_0, U_1) = 0$.

Then $(db)(U_0, U_1, \ldots, U_k) =$

$$\Big\{ b(U_1, U_2, \ldots, U_k) + b(U_0, U_2, \ldots, U_k) + \sum_{i=2}^{k} b(U_0, U_1, \ldots, \widehat{U}_i, \ldots, U_k) \Big\} \Big|_{U_0 \cap U_1 \cap \cdots \cap U_k} .$$

Since c was a cocycle, we have that

$$\{\sum_{i=2}^{k} b(U_0, U_1, \dots, \hat{U}_i, \dots, U_k)\}\big|_{U_0 \cap U_1 \cap \dots \cap U_k} = 0 \quad \text{if } \Omega(U_0, U_1) = 0;$$
$$= c(U_0, U_2, \dots, U_k)|_{U_0 \cap U_1 \cap \dots \cap U_k} + c(U_1, U_2, \dots, U_k)|_{U_0 \cap U_1 \cap \dots \cap U_k}$$

if $\Omega(U_0, U_1) = 1$.

We can compute the value of $(db)(U_0, U_1, \dots, U_k)$ for the four basic cases as in the computation of $(d \circ K)c$. This yields:

$$(db)(U_0, U_1, \dots, U_k) = 0 \qquad \text{in case (i).}$$
$$= c(U_1, U_2, \dots, U_k)|_{U_0 \cap U_1 \cap \dots \cap U_k} + c(U_0, U_2, \dots, U_k)|_{U_0 \cap U_1 \cap \dots \cap U_k} \qquad \text{in case (ii).}$$
$$= c(U_1, U_2, \dots, U_k)|_{U_0 \cap U_1 \cap \dots \cap U_k} \qquad \text{in case (iii).}$$
$$= c(U_0, U_2, \dots, U_k)|_{U_0 \cap U_1 \cap \dots \cap U_k} \qquad \text{in case (iv).}$$

Now the value of the cochain $(d \circ K)c + db$ is given by:

$$((d \circ K)c + db)(U_0, U_1, \dots, U_k) = 0 \qquad \text{in case (i)}$$
$$= 0 \qquad \text{in case (ii)}$$
$$= c(U_1, U_2, \dots, U_k)|_{U_0 \cap U_1 \cap \dots \cap U_k} \qquad \text{in case (iii)}$$
$$= c(U_1, U_2, \dots, U_k)|_{U_0 \cap U_1 \cap \dots \cap U_k} \qquad \text{in case (iv)}$$

On the other hand, $[\Omega] \cup [c]$ is represented by:

$$(\Omega \cup c)(U_0, U_1, \dots, U_k) = \Omega(U_0, U_1) \cdot c(U_1, U_2, \dots, U_k)|_{U_0 \cap U_1 \cap \dots \cap U_k},$$

and this is given by

$$
\begin{aligned}
(\Omega \cup c)(U_0, U_1, \dots, U_k) &= 0 && \text{in case (i)} \\
&= 0 && \text{in case (ii)} \\
&= c(U_1, U_2, \dots, U_k)\big|_{U_0 \cap U_1 \cap \cdots \cap U_k} && \text{in case (iii)} \\
&= c(U_1, U_2, \dots, U_k)\big|_{U_0 \cap U_1 \cap \cdots \cap U_k} && \text{in case (iv)}
\end{aligned}
$$

Thus we have derived $\Omega \cup c = (d \circ K)c + db$, i.e. $[\Omega] \cup [c] = \mu[c]$ on cohomology level, where μ is the connecting morphism of the Smith sequence.

Appendix B

Let p be a prime.

Let G be a finite group acting on a space X.

Suppose Y is the quotient space with respect to this action and $p \colon X \to Y$ is the projection.

Let $\mathscr{B}$ be a sheaf of $\mathbb{Z}_p$ - algebras on Y.

Then, as in **[Bredon 1]**, we have the following facts.

B1 FACT

For every $g \in G$ there is a unique g - cohomomorphism $\underline{g}$ such that the following diagram of cohomomorphisms commutes.

$$
\begin{tikzcd}
\overleftarrow{p}\mathscr{B} \arrow[rr, "\underline{g}"'] & & \overleftarrow{p}\mathscr{B} \\
& \mathscr{B} \arrow[ul, "p^*"] \arrow[ur, "p^*"'] &
\end{tikzcd}
$$

In turn, $\underline{g}$ uniquely determines a sheaf homomorphism $\gamma(g)$ such that the following diagram of cohomomorphisms commutes.

$$
\begin{tikzcd}
\overleftarrow{p}\mathscr{B} \arrow[r, "\underline{g}"'] & \overleftarrow{p}\mathscr{B} \\
\overleftrightarrow{p}\mathscr{B} \arrow[u, "p^+"] \arrow[r] & \overleftrightarrow{p}\mathscr{B} \arrow[u, "p^+"']
\end{tikzcd}
$$

$\gamma(g)$

Let $\nabla : \mathscr{B} \rightarrowtail \overleftrightarrow{p}\mathscr{B}$ be the natural inclusion given by the diagonal map stalkwise. Then $\mathscr{B}$ can be characterized as the fixed point set of the system $\{\gamma(g) \mid g \in G\}$.

B2 FACT

Let $\sigma : \overleftrightarrow{p}\mathscr{B} \to \overleftrightarrow{p}\mathscr{B}$ be given by $\sigma := \sum_{g \in G} \gamma(g)$.

Evidently, σ maps onto the subsheaf $\mathscr{B}$ in $\overleftrightarrow{p}\mathscr{B}$, i.e. $\sigma : \overleftrightarrow{p}\mathscr{B} \to \mathscr{B}$.

The cohomomorphism p^+ induces an isomorphism $p^+ : H^*(Y, \overleftrightarrow{p}\mathscr{B}) \to H^*(X, \overleftarrow{p}\mathscr{B})$ on cohomology level.

We can define $\Delta : H^*(X, \overleftarrow{p}\mathscr{B}) \to H^*(Y, \mathscr{B})$ by $\Delta := \sigma \circ (p^+)^{-1}$; Δ is called the transfer map.

Let $T : H^*(X, \overleftarrow{p}\mathscr{B}) \to H^*(X, \overleftarrow{p}\mathscr{B})$ be defined by $T := \sum_{g \in G} \underline{g}$.

Let $H^*(X, \overleftarrow{p}\mathscr{B})^G$ be the subgroup of all those elements in $H^*(X, \overleftarrow{p}\mathscr{B})$ that are invariant to the system $\{\underline{g} \mid g \in G\}$. Evidently, $T : H^*(X, \overleftarrow{p}\mathscr{B}) \to H^*(X, \overleftarrow{p}\mathscr{B})^G$. Thus we have the equality $T = p^* \circ \Delta$. Further, p^* obviously maps $H^*(Y, \mathscr{B})$ into the subgroup $H^*(X, \overleftarrow{p}\mathscr{B})^G$.

B3 FACT

Suppose further that p does not divide the order of G. Under this condition we have that the map $p^* : H^*(Y, \mathscr{B}) \to H^*(X, \overleftarrow{p}\mathscr{B})^G$ is an isomorphism.

B4 FACT

Let X be a topological space and $\mathscr{A}$ be a sheaf of $\mathbb{Z}_p$ - algebras on X. Then there is a system of maps $\left\langle St^j : H^*(X, \mathscr{A}) \to H^{*+j}(X, \mathscr{A}) \mid j = 0, 1, \ldots \right\rangle$ called the Steenrod p - powers. Each St^j is natural with respect to cohomomorphisms of $\mathbb{Z}_p$ - algebras.

From these facts we can easily deduce the following theorem.

B5 THEOREM

Let G be a finite group acting on X.

Let $p: X \to Y$ be the projection to the quotient space.

Let $\mathscr{B}$ be a sheaf of $\mathbb{Z}_p$ - algebras on Y and suppose that p does not divide the order of G.

Equip $\overleftarrow{p}\mathscr{B}$ with the pullback $\mathbb{Z}_p$ - algebra structure.

Let Δ be the transfer map.

Then the following diagram commutes.

$$\begin{tikzcd}
H^*(X, \overleftarrow{p}\mathscr{B}) \arrow[r, "St^j"] \arrow[d, "\Delta"'] & H^*(X, \overleftarrow{p}\mathscr{B}) \arrow[d, "\Delta"] \\
H^*(Y, \mathscr{B}) \arrow[r, "St^j"'] & H^*(Y, \mathscr{B})
\end{tikzcd}$$

Proof:

Our commutator is represented as one of the faces of the diagram on the following page.

$$
\begin{tikzcd}
 & H^*(X, \overleftarrow{p}\mathscr{S}) \arrow[dl, "T"'] \arrow[dd, "\Delta"] & & H^*(X, \overleftarrow{p}\mathscr{S}) \arrow[ll, "\mathrm{St}^j"'] \arrow[dr, "T"] \arrow[dd, "\Delta"'] & \\
H^*(X, \overleftarrow{p}\mathscr{S})^G & & & & H^*(X, \overleftarrow{p}\mathscr{S})^G \arrow[llll, "\mathrm{St}^j"'] \\
 & H^*(Y, \mathscr{S}) \arrow[ul, "p^*"] & & H^*(Y, \mathscr{S}) \arrow[ll, "\mathrm{St}^j"'] \arrow[ur, "p^*"'] &
\end{tikzcd}
$$

By the naturality of St^j with respect to cohomomorphisms and by $T = p^* \circ \Delta$ we know that all the remaining faces are commutative. Hence the front face must be commutative also. ■

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